SOLUTIONS MANUAL

SIGNALS & SYSTEMS

SECOND EDITION

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Chapter 1 Answers

1.1. Converting from polar to Cartesian coordinates:

$\frac{1}{2}e^{j\pi} = \frac{1}{2}\cos\pi = -\frac{1}{2}$, $\quad \frac{1}{2}e^{-j\pi} = \frac{1}{2}\cos(-\pi) = -\frac{1}{2}$

$e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$, $\quad e^{-j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$

$e^{j5\frac{\pi}{2}} = e^{j\frac{\pi}{2}} = j$, $\quad \sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) = 1+j$

$\sqrt{2}e^{\frac{9j\pi}{4}} = \sqrt{2}e^{\frac{j\pi}{4}} = 1+j$, $\quad \sqrt{2}e^{\frac{-9j\pi}{4}} = \sqrt{2}e^{\frac{-j\pi}{4}} = 1-j$

$\sqrt{2}e^{\frac{-j\pi}{4}} = 1-j$

1.2. Converting from Cartesian to polar coordinates:

$5 = 5e^{j0}$, $\quad -2 = 2e^{j\pi}$, $\quad -3j = 3e^{-j\frac{\pi}{2}}$

$\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{\pi}{3}}$, $\quad 1+j = \sqrt{2}e^{j\frac{\pi}{4}}$, $\quad (1-j)^2 = 2e^{-j\frac{\pi}{2}}$

$j(1-j) = e^{j\frac{\pi}{4}}$, $\quad \frac{1+j}{1-j} = e^{j\frac{\pi}{2}}$, $\quad \frac{\sqrt{2}+j\sqrt{2}}{1+j\sqrt{3}} = e^{-j\frac{\pi}{12}}$

1.3. (a) $E_\infty = \int_0^\infty e^{-4t}dt = \frac{1}{4}$, $P_\infty = 0$, because $E_\infty < \infty$

(b) $x_2(t) = e^{j(2t+\frac{\pi}{4})}$, $|x_2(t)| = 1$. Therefore, $E_\infty = \int_{-\infty}^{\infty}|x_2(t)|^2dt = \int_{-\infty}^{\infty}dt = \infty$, $P_\infty =$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x_2(t)|^2dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt = \lim_{T\to\infty} 1 = 1$$

(c) $x_3(t) = \cos(t)$. Therefore, $E_\infty = \int_{-\infty}^{\infty}|x_3(t)|^2dt = \int_{-\infty}^{\infty}\cos^2(t)dt = \infty$,

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\frac{1+\cos(2t)}{2}\right)dt = \frac{1}{2}$$

(d) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$, $|x_1[n]|^2 = \left(\frac{1}{4}\right)^n u[n]$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty}|x_1[n]|^2 = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{4}{3}$.

$P_\infty = 0$, because $E_\infty < \infty$.

(e) $x_2[n] = e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}$, $|x_2[n]|^2 = 1$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty}|x_2[n]|^2 = \infty$,

$$P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x_2[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}1 = 1.$$

(f) $x_3[n] = \cos(\frac{\pi}{4}n)$. Therefore, $E_\infty = \sum_{n=-\infty}^{\infty}|x_3[n]|^2 = \sum_{n=-\infty}^{\infty}\cos^2(\frac{\pi}{4}n) = \infty$,

$$P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\cos^2(\frac{\pi}{4}n) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\left(\frac{1+\cos(\frac{\pi}{2}n)}{2}\right) = \frac{1}{2}$$

1.4. (a) The signal $x[n]$ is shifted by 3 to the right. The shifted signal will be zero for $n < 1$ and $n > 7$.

(b) The signal $x[n]$ is shifted by 4 to the left. The shifted signal will be zero for $n < -6$ and $n > 0$.

(c) The signal $x[n]$ is flipped. The flipped signal will be zero for $n < -4$ and $n > 2$.

(d) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the right. This new signal will be zero for $n < -2$ and $n > 4$.

(e) The signal $x[n]$ is flipped and the flipped signal is shifted by 2 to the left. This new signal will be zero for $n < -6$ and $n > 0$.

1.5. (a) $x(1-t)$ is obtained by flipping $x(t)$ and shifting the flipped signal by 1 to the right. Therefore, $x(1-t)$ will be zero for $t > -2$.

(b) From (a), we know that $x(1-t)$ is zero for $t > -2$. Similarly, $x(2-t)$ is zero for $t > -1$. Therefore, $x(1-t) + x(2-t)$ will be zero for $t > -2$.

(c) $x(3t)$ is obtained by linearly compressing $x(t)$ by a factor of 3. Therefore, $x(3t)$ will be zero for $t < 1$.

(d) $x(t/3)$ is obtained by linearly stretching $x(t)$ by a factor of 3. Therefore, $x(t/3)$ will be zero for $t < 9$.

1.6. (a) $x_1(t)$ is not periodic because it is zero for $t < 0$.

(b) $x_2[n] = 1$ for all n. Therefore, it is periodic with a fundamental period of 1.

(c) $x_3[n]$ is as shown in the Figure S1.6.

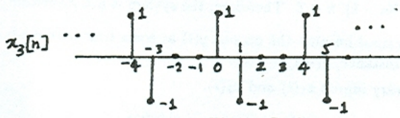

Figure S1.6

Therefore, it is periodic with a fundamental period of 4.

1.7. (a)

$$\mathcal{E}v\{x_1[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}(u[n] - u[n-4] + u[-n] - u[-n-4])$$

Therefore, $\mathcal{E}v\{x_1[n]\}$ is zero for $|n| > 3$.

(b) Since $x_2(t)$ is an odd signal, $\mathcal{E}v\{x_2(t)\}$ is zero for all values of t.

(c)

$$\mathcal{E}v\{x_3[n]\} = \frac{1}{2}(x_1[n] + x_1[-n]) = \frac{1}{2}[(\tfrac{1}{2})^n u[n-3] - (\tfrac{1}{2})^{-n}u[-n-3]]$$

Therefore, $\mathcal{E}v\{x_3[n]\}$ is zero when $|n| < 3$ and when $|n| \to \infty$.

(d)

$$\mathcal{E}v\{x_4(t)\} = \frac{1}{2}(x_4(t) + x_4(-t)) = \frac{1}{2}[e^{-5t}u(t+2) - e^{5t}u(-t+2)]$$

Therefore, $\mathcal{E}v\{x_4(t)\}$ is zero only when $|t| \to \infty$.

1.8. (a) $\mathcal{R}e\{x_1(t)\} = -2 = 2e^{0t}\cos(0t+\pi)$

(b) $\mathcal{R}e\{x_2(t)\} = \sqrt{2}\cos(\frac{\pi}{4})\cos(3t+2\pi) = \cos(3t) = e^{0t}\cos(3t+0)$

(c) $\mathcal{R}e\{x_3(t)\} = e^{-t}\sin(3t+\pi) = e^{-t}\cos(3t+\frac{\pi}{2})$

(d) $\mathcal{R}e\{x_4(t)\} = -e^{-2t}\sin(100t) = e^{-2t}\sin(100t+\pi) = e^{-2t}\cos(100t+\frac{\pi}{2})$

1.9. (a) $x_1(t)$ is a periodic complex exponential.

$$x_1(t) = je^{j10t} = e^{j(10t+\frac{\pi}{2})}$$

The fundamental period of $x_1(t)$ is $\frac{2\pi}{10} = \frac{\pi}{5}$.

(b) $x_2(t)$ is a complex exponential multiplied by a decaying exponential. Therefore, $x_2(t)$ is not periodic.

(c) $x_3[n]$ is a periodic signal.

$$x_3[n] = e^{j7\pi n} = e^{j\pi n}$$

$x_3[n]$ is a complex exponential with a fundamental period of $\frac{2\pi}{\pi} = 2$.

(d) $x_4[n]$ is a periodic signal. The fundamental period is given by $N = m(\frac{2\pi}{3\pi/5}) = m(\frac{10}{3})$. By choosing $m = 3$, we obtain the fundamental period to be 10.

(e) $x_5[n]$ is not periodic. $x_5[n]$ is a complex exponential with $\omega_0 = 3/5$. We cannot find any integer m such that $m(\frac{2\pi}{\omega_0})$ is also an integer. Therefore, $x_5[n]$ is not periodic.

1.10.

$$x(t) = 2\cos(10t+1) - \sin(4t-1)$$

Period of first term in RHS $= \frac{2\pi}{10} = \frac{\pi}{5}$

Period of second term in RHS $= \frac{2\pi}{4} = \frac{\pi}{2}$

Therefore, the overall signal is periodic with a period which is the least common multiple of the periods of the first and second terms. This is equal to π.

1.11.

$$x[n] = 1 + e^{j\frac{4\pi}{7}n} - e^{j\frac{2\pi}{5}n}$$

Period of the first term in the RHS = 1

Period of the second term in the RHS $= m(\frac{2\pi}{4\pi/7}) = 7$ (when $m = 2$)

Period of the third term in the RHS $= m(\frac{2\pi}{2\pi/5}) = 5$ (when $m = 1$)

Therefore, the overall signal $x[n]$ is periodic with a period which is the least common multiple of the periods of the three terms in $x[n]$. This is equal to 35.

1.12. The signal $x[n]$ is as shown in Figure S1.12. $x[n]$ can be obtained by flipping $u[n]$ and then shifting the flipped signal by 3 to the right. Therefore, $x[n] = u[-n+3]$. This implies that $M = -1$ and $n_0 = -3$.

Figure S1.12

1.13.

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t}(\delta(\tau+2) - \delta(\tau-2))d\tau = \begin{cases} 0, & t < -2 \\ 1, & -2 \le t \le 2 \\ 0, & t > 2 \end{cases}$$

Therefore,

$$E_\infty = \int_{-2}^{2} dt = 4$$

1.14. The signal $x(t)$ and its derivative $g(t)$ are shown in Figure S1.14.

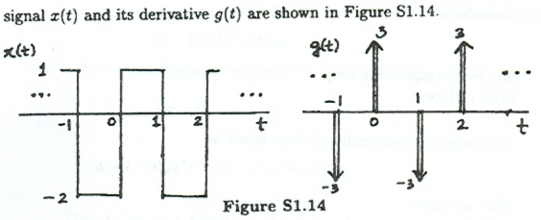

Figure S1.14

Therefore,

$$g(t) = 3\sum_{k=-\infty}^{\infty}\delta(t-2k) - 3\sum_{k=-\infty}^{\infty}\delta(t-2k-1)$$

This implies that $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

1.15. (a) The signal $x_2[n]$, which is the input to S_2, is the same as $y_1[n]$. Therefore,

$$\begin{aligned} y_2[n] &= x_2[n-2] + \frac{1}{2}x_2[n-3] \\ &= y_1[n-2] + \frac{1}{2}y_1[n-3] \\ &= 2x_1[n-2] + 4x_1[n-3] + \frac{1}{2}(2x_1[n-3] + 4x_1[n-4]) \\ &= 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4] \end{aligned}$$

The input-output relationship for S is

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

(iii) In this case
$$X(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}.$$
Therefore,
$$Y(e^{j\omega}) = 1.$$
Taking the inverse Fourier transform, we obtain
$$y[n] = \delta[n].$$

(iv) In this case
$$X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$
Therefore,
$$\begin{aligned} Y(e^{j\omega}) &= \left[1 - \frac{1}{2}e^{-j\omega}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= -1 + \frac{2}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$
Taking the inverse Fourier transform, we obtain
$$y[n] = -\delta[n] + 2\left(-\frac{1}{2}\right)^n u[n].$$

(c) (i) We have
$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{(1+\frac{1}{2}e^{-j\omega})^2} - \frac{\frac{1}{4}e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})^2} \end{aligned}$$
Taking the inverse Fourier transform, we obtain
$$y[n] = (n+1)\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4}n\left(-\frac{1}{2}\right)^{n-1} u[n-1].$$

(ii) We have
$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{1-\frac{1}{4}e^{-j\omega}} \end{aligned}$$
Taking the inverse Fourier transform, we obtain
$$y[n] = \left(\frac{1}{4}\right)^n u[n].$$

(iii) We have
$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{2/3}{(1+\frac{1}{2}e^{-j\omega})^2} + \frac{2/9}{1+\frac{1}{2}e^{-j\omega}} + \frac{1/9}{1-\frac{1}{4}e^{-j\omega}} \end{aligned}$$
Taking the inverse Fourier transform, we obtain
$$y[n] = \frac{2}{3}(n+1)\left(-\frac{1}{2}\right)^n u[n] + \frac{2}{9}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{9}\left(\frac{1}{4}\right)^n u[n].$$

(iv) We have
$$\begin{aligned} Y(e^{j\omega}) &= \left[1 + 2e^{-3j\omega}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{1+\frac{1}{2}e^{-j\omega}} + \frac{2e^{-3j\omega}}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$
Taking the inverse Fourier transform, we obtain
$$y[n] = \left(-\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{2}\right)^{n-3} u[n-3].$$

5.34. (a) Since the two systems are cascaded, the frequency response of the overall system is
$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\ &= \frac{2 - e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}} \end{aligned}$$
Therefore, the Fourier transforms of the input and output of the overall system are related by
$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}}.$$
Cross-multiplying and taking the inverse Fourier transform, we get
$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1].$$

(b) We may rewrite the overall frequency response as
$$H(e^{j\omega}) = \frac{4/3}{1+\frac{1}{2}e^{-j\omega}} + \frac{(1+j\sqrt{3})/3}{1-\frac{1}{2}e^{j120}e^{-j\omega}} + \frac{(1-j\sqrt{3})/3}{1-\frac{1}{2}e^{-j120}e^{-j\omega}}.$$
Taking the inverse Fourier transform we get
$$h[n] = \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1+j\sqrt{3}}{3}\left(\frac{1}{2}e^{j120}\right)^n u[n] + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j120}\right)^n u[n].$$

5.35. (a) Taking the Fourier transform of both sides of the given difference equation we obtain
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b + e^{-j\omega}}{1 - ae^{-j\omega}}.$$
In order for $|H(e^{j\omega})|$ to be one, we must ensure that
$$\begin{aligned} |b + e^{-j\omega}| &= |1 - ae^{-j\omega}| \\ 1 + b^2 + 2b\cos\omega &= 1 + a^2 - 2a\cos\omega \end{aligned}$$
This is possible only if $b = -a$.

(b) The plot is as shown Figure S5.35.

(c) The plot is as shown Figure S5.35.

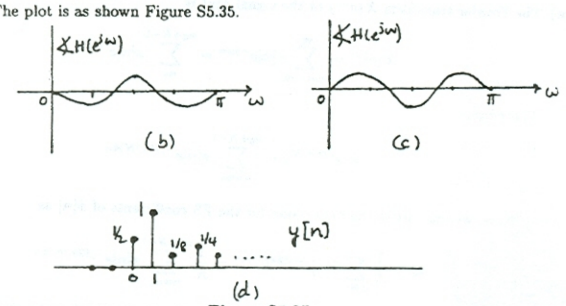

Figure S5.35

(d) When $a = -\frac{1}{2}$,
$$H(e^{j\omega}) = \frac{\frac{1}{2} + e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}.$$
Also,
$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}.$$
Therefore,
$$\begin{aligned} Y(e^{j\omega}) &= \frac{\frac{1}{2} + e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} \\ &= \frac{5/4}{1-\frac{1}{2}e^{-j\omega}} - \frac{3/4}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$
Taking the inverse Fourier transform we obtain
$$y[n] = \frac{5}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{4}\left(-\frac{1}{2}\right)^n u[n].$$

This is as sketched in Figure S5.35.

5.36. (a) The frequency responses are related by the following expression:
$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}.$$

(b) (i) Here, $H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$. Therefore, $G(e^{j\omega}) = 1/(1-\frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n]$. Since
$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{4}e^{-j\omega}},$$
the difference equation relating the input $x[n]$ and output $y[n]$ is
$$y[n] - \frac{1}{4}y[n-1] = x[n].$$

(ii) Here, $H(e^{j\omega}) = 1/(1+\frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$ and $g[n] = \delta[n] + \frac{1}{2}\delta[n-1]$. Since
$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{2}e^{-j\omega},$$
the difference equation relating the input $x[n]$ and output $y[n]$ is
$$y[n] = x[n] + \frac{1}{2}x[n-1].$$

(iii) Here, $H(e^{j\omega}) = (1-\frac{1}{4}e^{-j\omega})/(1+\frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{1}{2}e^{-j\omega})/(1-\frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n] + \frac{1}{2}(\frac{1}{4})^{n-1}u[n-1]$. Since
$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}},$$
the difference equation relating the input $x[n]$ and output $y[n]$ is
$$y[n] - \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1].$$

(iv) Here, $H(e^{j\omega}) = (1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})/(1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})/(1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})$. Therefore,
$$G(e^{j\omega}) = 1 + \frac{2}{1-(1/2)e^{-j\omega}} - \frac{2}{1+(1/4)e^{-j\omega}}$$
and
$$g[n] = \delta[n] + 2\left(\frac{1}{2}\right)^n u[n] - 2\left(-\frac{1}{4}\right)^n u[n].$$
Since
$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})}{(1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})},$$
the difference equation relating the input $x[n]$ and output $y[n]$ is
$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-1] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(g) The unilateral Laplace transform of $x[n] = 2^n u[-n] + (1/4)^n u[n-1]$ is

$$\begin{aligned}\mathcal{X}(z) &= \sum_{n=0}^{\infty} 2^n u[-n] + (1/4)^n u[n-1] z^{-n} \\ &= \sum_{n=0}^{\infty} (1/4)^n z^{-n} \\ &= \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{All } z.\end{aligned}$$

(h) The unilateral Laplace transform of $x[n] = (1/3)^{n-2} u[n-2]$ is

$$\begin{aligned}\mathcal{X}(z) &= \sum_{n=0}^{\infty} (1/3)^{n-2} u[n-2] z^{-n} \\ &= z^{-2} \sum_{n=0}^{\infty} (1/3)^n z^{-n} \\ &= \frac{z^{-2}}{1-(1/2)z^{-1}}, \quad |z| > 1/2.\end{aligned}$$

10.41. From the given information,

$$\begin{aligned}\mathcal{X}_1(z) &= \sum_{n=0}^{\infty} (1/2)^{n+1} u[n+1] z^{-n} \\ &= (1/2) \sum_{n=0}^{\infty} (1/2)^n z^{-n} \\ &= \frac{1/2}{1-(1/2)z^{-1}}, \quad |z| > 1/2\end{aligned}$$

and

$$\begin{aligned}\mathcal{X}_2(z) &= \sum_{n=0}^{\infty} (1/4)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} (1/4)^n z^{-n} \\ &= \frac{1}{1-(1/4)z^{-1}}, \quad |z| > 1/4.\end{aligned}$$

Using Table 10.2 and the time shift property we get

$$X_1(z) = \frac{z}{1-\frac{1}{2}z^{-1}}, \quad |z| > 1/2.$$

and

$$X_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > 1/4.$$

(a) We have

$$G(z) = X_1(z) X_2(z) = \frac{z}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}.$$

The ROC is $|z| > (1/2)$. The partial fraction expansion of $G(z)$ is

$$G(z) = z\left[\frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right].$$

Using Table 10.2 and the time shift property, we get

$$g[n] = 2\left(\frac{1}{2}\right)^{n+1} u[n+1] - \left(\frac{1}{4}\right)^{n+1} u[n+1].$$

(b) We have

$$\mathcal{Q}(z) = \mathcal{X}_1(z)\mathcal{X}_2(z) = \frac{1/2}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}.$$

The ROC of $\mathcal{Q}(z)$ is $|z| > (1/2)$. The partial fraction expansion of $\mathcal{Y}(z)$ is

$$\mathcal{Q}(z) = \frac{1}{2}\left[\frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right].$$

Therefore,

$$q[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{4}\right)^n u[n].$$

Clearly, $q[n] \neq g[n]$ for $n > 0$.

10.42. (a) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) + 3z^{-1}\mathcal{Y}(z) + 3y[-1] = \mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = \frac{-3}{1+3z^{-1}}.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = -3(-3)^n u[n] = (-3)^{n+1} u[n].$$

Now, since it is given that $x[n] = (1/2)^n u[n]$, we have

$$\mathcal{X}(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > 1/2.$$

Setting $y[-1]$ to be zero, we get

$$\mathcal{Y}(z) + 3z^{-1}\mathcal{Y}(z) = \frac{1}{1-\frac{1}{2}z^{-1}}.$$

Therefore,

$$\mathcal{Y}(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}.$$

The partial fraction expansion of $\mathcal{Y}(z)$ is

$$\mathcal{Y}(z) = \frac{1/7}{1-\frac{1}{2}z^{-1}} + \frac{6/7}{1+3z^{-1}}.$$

The inverse unilateral z-transform gives the zero-state response

$$y_{zs}[n] = \frac{1}{7}\left(\frac{1}{2}\right)^n u[n] + \frac{6}{7}(-3)^n u[n].$$

(b) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = 0.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = 0.$$

Now, since it is given that $x[n] = u[n]$, we have

$$\mathcal{X}(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1.$$

Setting $y[-1]$ to be zero, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) = \frac{1}{1-z^{-1}} - \frac{(1/2)z^{-1}}{1-z^{-1}}.$$

Therefore,

$$\mathcal{Y}(z) = \frac{1}{1-z^{-1}}.$$

The inverse unilateral z-transform gives the zero-state response

$$y_{zs}[n] = u[n].$$

(c) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = \frac{1/2}{1-\frac{1}{2}z^{-1}}.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = \left(\frac{1}{2}\right)^{n+1} u[n].$$

Since the input $x[n]$ is the same as the one used in the part (b), the zero-state response is still

$$y_{zs}[n] = u[n].$$

10.43. (a) First let us determine the z-transform $X_1(z)$ of the sequence $x_1[n] = x[-n]$ in terms of $X(z)$:

$$\begin{aligned}X_1(z) &= \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n] z^{n} \\ &= X(1/z)\end{aligned}$$

Therefore, if $x[n] = x[-n]$, then $X(z) = X(1/z)$.

(b) If z_0 is a pole, then $1/X(z_0) = 0$. From the result of part (a), we know that $X(z_0) = X(1/z_0)$. Therefore, $1/X(z_0) = 1/X(1/z_0) = 0$. This implies that there is a pole at $1/z_0$.

If z_0 is a zero, then $X(z_0) = 0$. From the result of part (a), we know that $X(z_0) = X(1/z_0) = 0$. This implies that there is a zero at $1/z_0$.

(c) (1) In this case,

$$X(z) = z + z^{-1} = \frac{1+z^2}{z}, \quad |z| > 0.$$

$X(z)$ has zeros $z_1 = j$ and $z_2 = -j$. Also, $X(z)$ has the poles $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = 1/z_1$ and $p_1 = 1/p_2$, which proves that the statement of (b) is true.

(2) In this case,

$$X(z) = z - \frac{5}{2} + z^{-1} = \frac{1-\frac{5}{2}z+z^2}{z}, \quad |z| > 0.$$

$X(z)$ has zeros $z_1 = -1/2$ and $z_2 = -2$. Also, $X(z)$ has the poles $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = 1/z_1$ and $p_1 = 1/p_2$, which proves that the statement of (b) is true.

10.44. (a) Using the shift property, we get

$$\mathcal{Z}\{\Delta x[n]\} = X(z) - z^{-1}X(z) = (1-z^{-1})X(z).$$

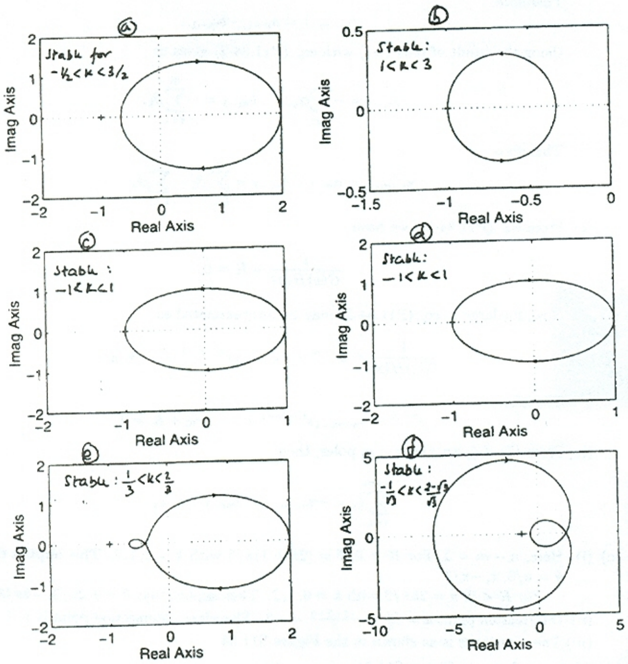

Figure S11.30

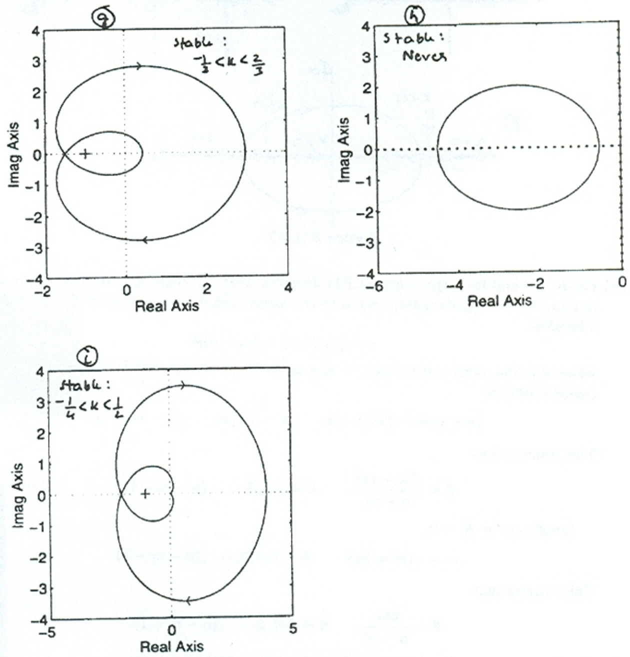

Figure S11.30

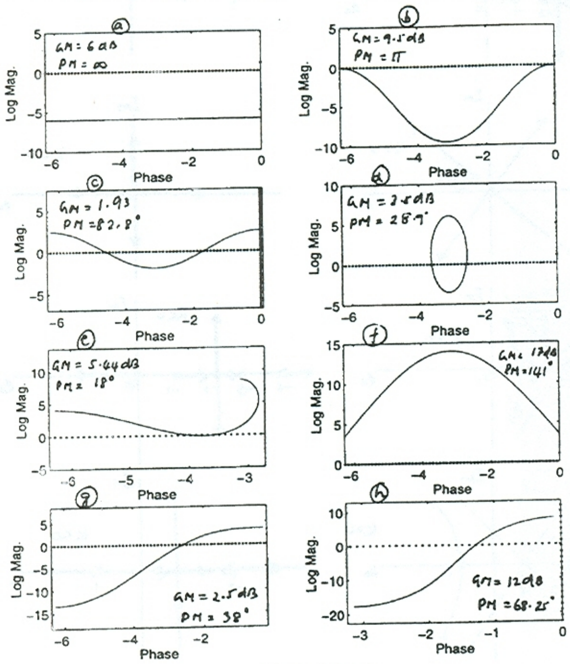

Figure S11.31

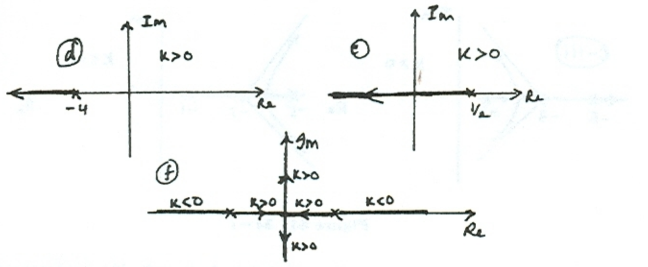

Figure S11.32

(d) In this case

$$Q(s) = \frac{s+1}{s+2}\left[\frac{1}{s+k+4}\right].$$

The zero which is independent of K is at $s = -1$. The pole which is independent of K is at $s = -2$. The root locus for the remaining closed-loop pole is as shown in Figure S11.32 for $K > 0$.

(e) In this case

$$Q(z) = (z+1)\left[\frac{1}{z+k-(1/2)}\right].$$

The zero which is independent of K is at $z = -1$. The root locus for the pole is as shown in Figure S11.32 for $K > 0$.

(f) (i) For this case, we have $G(z)H(z) = 1/[(z-2)(z+2)]$. The root-locus for $K > 0$ and $K < 0$ are shown in Figure S11.32.

(ii) The system is stable for when the closed-loop poles are within the unit circle. The closed-loop poles satisfy the condition

$$G(z)H(z) = -1/K.$$

Therefore, looking at the plots from before, it is clear that as K increases, the system becomes stable when $G(1)H(1) = -1/K$. That is, the system becomes stable when $K > 3$. As K continues to increase, the system again becomes unstable when $G(j)H(j) = -1/K$. That is, the system becomes unstable when $K > 5$. Therefore, the system is stable for $3 < K < 5$.

(iii) When $K = 4$, $Q(z) = 1$. Therefore, $q[n] = \delta[n]$.

11.33. The root loci are as shown in Figure S11.33.