SOLUTIONS MANUAL

Logic and Discrete Mathematics

A Concise Introduction

WILLEM CONRADIE
VALENTIN GORANKO
CLAUDETTE ROBINSON

WILEY

Logic and Discrete Mathematics

Logic and Discrete Mathematics

A Concise Introduction, Solutions Manual

Willem Conradie
University of Johannesburg, South Africa

Valentin Goranko
Stockholm University, Sweden

Claudette Robinson
University of Johannesburg, South Africa

WILEY

This edition first published 2015

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com.

Library of Congress Cataloging-in-Publication Applied Data for.

A catalogue record for this book is available from the British Library.

ISBN: 9781118762677

Set in 9/11pt, PalatinoLTStd by Laserwords Private Limited, Chennai, India

1 2015

Contents

Preface

This manual contains answers and solutions to roughly three quarters of the exercises in *Logic and Discrete Mathematics: A Concise Introduction* by Willem Conradie and Valentin Goranko. Most solutions are worked out in full detail. In deciding which solutions to include we were guided by two principles: fundamental exercises were given preference above the more esoteric ones intended mainly for enrichment; where a number of very similar exercises occur in succession, complete solutions were given for a few while the others were omitted or provided with answers only. We trust that these solutions will be a very valuable resource to students and instructors using *Logic and Discrete Mathematics*.

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/conradie/logic

The website includes:

- Lecture Slides
- Quizzes

1

Preliminaries

1.1. Sets

(1a) $A = \{-1, 4\}$

(1b) $B = \{-5, -3, -\frac{3}{2}, 1\}$

(1c) $C = \{-3, -2, -1, 0, 1, 2\}$

(1d) $D = \{-1\}$

(1e) $E = \{-3, -2, 0, 1, 2\}$

(1f) $F = \{1, 3, 5, 7, 9\}$

(2a) $B = \{x \in \mathbb{R} \mid x^2 - 3 = 0\}$

(2b) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z} \text{ and } 1 \leq k \leq 4\}$

(2c) $A = \{x \mid x = 2k, \text{ where } k \in \mathbb{Z}^+\}$

(2d) $A = \{x \mid x = 3k + 2, \text{ where } k \in \mathbb{Z}\}$

(2e) $A = \{x \mid x = n^3, \text{ where } n \in \mathbb{N}\}$

(2f) $B = \{x \in \mathbb{Z} \mid -3 \leq x \leq 6\}$

(2g) $B = \{x \in \mathbb{Z} \mid x \leq -3 \vee x \geq 5\}$

(3) Only a, d e, g, h and j are true.

(4) Only a, g, j and l are true.

(5a) $A \cap B = \{2, 7\}$

(5b) $A \cup B = \{0, 1, 2, 4, 5, 6, 7, 9\}$

(5c) $A - B = \{1, 4\}$

(5d) $B - A = \{0, 5, 6, 9\}$

(5e) $A - (A - B) = \{2, 7\}$

(5f) $B - (A - B) = B$

(5g) $B' = \{1, 3, 4, 8\}$

(5h) $(A \cap B)' = \{0, 1, 3, 4, 5, 6, 8, 9\}$

(5i) $A \cap B' = \{1, 4\}$

(5j) $(A \cup B)' = \{3, 8\}$

(5k) $(A \cup B)' - (B - A)' = \varnothing$

(6a) $A \cap B = \{0\}$

(6b) $A \cup B = \{0, A\}$

(6c) $B - A = \{A\}$

(6d) $A \cap (B \cup C) = \{0\}$

(6e) $(B \cup C) - A = \{A, \{0\}, \{A\}\}$

(6f) $(A \cap B) \cup (A \cap C) = \{0\}$

(6g) $(C - B) - A = \{\{0\}, \{A\}\}$

(6h) $A \cap (C - A) = \varnothing$

Logic and Discrete Mathematics: A Concise Introduction, Solutions Manual, First Edition.
Willem Conradie, Valentin Goranko and Claudette Robinson.

Companion Website: www.wiley.com/go/conradie/logic

(3a) The tableau closes, so the consequence holds:

- $P \to Q : \mathsf{T}, Q \to R : \mathsf{T}, P \wedge S : \mathsf{T}, R \wedge S : \mathsf{F}$
  - $P : \mathsf{T}, S : \mathsf{T}$
    - $P : \mathsf{F}$ $\times$
    - $Q : \mathsf{T}$
      - $Q : \mathsf{F}$ $\times$
      - $R : \mathsf{T}$
        - $R : \mathsf{F}$ $\times$
        - $S : \mathsf{F}$ $\times$

(3b) The tableau closes, so the consequence holds:

- $R : \mathsf{F}, P \wedge \neg R : \mathsf{F}, \neg P \to \neg Q : \mathsf{T}, Q : \mathsf{T}$
  - $P : \mathsf{F}$
    - $\neg P : \mathsf{F}$
      - $P : \mathsf{T}$ $\times$
    - $\neg Q : \mathsf{T}$
      - $Q : \mathsf{F}$ $\times$
  - $\neg R : \mathsf{F}$
    - $R : \mathsf{T}$ $\times$

(3c) The tableau closes, so the consequence holds:

- $Q \to (R \wedge S) : \mathsf{T}, Q \wedge S : \mathsf{F}, R \wedge Q : \mathsf{T}$
  - $R : \mathsf{T}, Q : \mathsf{T}$
    - $Q : \mathsf{F}$ $\times$
    - $S : \mathsf{F}$
      - $Q : \mathsf{F}$ $\times$
      - $R \wedge S : \mathsf{T}$
        - $R : \mathsf{T}, S : \mathsf{T}$ $\times$

(7f) Using predicates $B(x)$ for "x is a bachelor", $M(x)$ for "x is a man", $W(x)$ for "x is a woman", $K(x, y)$ for "x knows y" and $L(x, y)$ for "x loves y", we can formalize the argument as follows:

$$\begin{array}{l}
\exists x(B(x) \wedge \forall y(W(y) \wedge K(x,y) \rightarrow L(x,y))) \\
\forall x(B(x) \rightarrow M(x)) \\
W(Eve) \wedge \forall x(M(x) \rightarrow K(x, Eve)) \\
\hline
\exists x(B(x) \wedge \exists y(W(y) \wedge L(x,y)))
\end{array}$$

The argument is logically valid:

$$\exists x(B(x) \wedge \forall y(W(y) \wedge K(x,y) \rightarrow L(x,y))) : \mathsf{T}, \forall x(B(x) \rightarrow M(x)) : \mathsf{T}$$
$$W(Eve) \wedge \forall x(M(x) \rightarrow K(x, Eve)) : \mathsf{T}, \exists x(B(x) \wedge \exists y(W(y) \wedge L(x,y))) : \mathsf{F}$$

- $B(c) \wedge \forall y(W(y) \wedge K(c,y) \rightarrow L(c,y)) : \mathsf{T}$
- $B(c) : \mathsf{T}, \forall y(W(y) \wedge K(c,y) \rightarrow L(c,y)) : \mathsf{T}$
- $W(Eve) : \mathsf{T}, \forall x(M(x) \rightarrow K(x, Eve)) : \mathsf{T}$
- $B(c) \wedge \exists y(W(y) \wedge L(c,y)) : \mathsf{F}$
  - $B(c) : \mathsf{F}$ $\times$
  - $\exists y(W(y) \wedge L(c,y)) : \mathsf{F}$
    - $W(Eve) \wedge L(c, Eve) : \mathsf{F}$
      - $W(Eve)) : \mathsf{F}$ $\times$
      - $L(c, Eve) : \mathsf{F}$
        - $W(Eve) \wedge K(c, Eve) \rightarrow L(c, Eve) : \mathsf{T}$
          - $W(Eve) \wedge K(c, Eve) : \mathsf{F}$
            - $W(Eve) : \mathsf{F}$ $\times$
            - $K(c, Eve) : \mathsf{F}$
              - $M(c) \rightarrow K(c, Eve) : \mathsf{T}$
                - $M(c) : \mathsf{F}$
                  - $B(c) \rightarrow M(c) : \mathsf{T}$
                    - $B(c) : \mathsf{F}$ $\times$
                    - $M(c) : \mathsf{T}$ $\times$
                - $K(c, Eve) : \mathsf{T}$ $\times$
          - $L(c, Eve) : \mathsf{T}$ $\times$

(9) We have $606 > 605 = (5)(121)$, so, by the CPHP, if we divide the square into a 11×11 grid so that each of the smaller squares has a side $\frac{1}{11}$, at least 6 points will end up in the same small square. The farthest apart any two points in this square can be is the length of its diagonal; that is, $\sqrt{\left(\frac{1}{11}\right)^2 + \left(\frac{1}{11}\right)^2} = \sqrt{\frac{1}{121} + \frac{1}{121}} = \sqrt{\frac{2}{121}} = \frac{\sqrt{2}}{11}$. Inscribe the square in a circle with diameter $\frac{\sqrt{2}}{11}$, i.e. a radius of $\frac{\sqrt{2}}{22}$. However, $\frac{1}{15} > \frac{\sqrt{2}}{22}$, so we can cover the six points with a circle of radius $\frac{1}{15}$.

(10a) Let P be the set of balls we select and H the set of different colours. To apply the CPHP, we need $|P| > (4)(4) = 16$, so choosing 17 or more balls will guarantee 5 balls of the same colour.

(10b) The worst case scenario is selecting all the blue, red and yellow balls without selecting any green balls. In total, we can select $13 + 10 + 6 = 29$ balls without selecting a green ball. Hence, we have to select at least 32 balls in order to make sure that there will be at least 3 green balls.

(10c) The worst case scenario is selecting all the blue, red and green balls without selecting a yellow ball. In total, we can select $13 + 10 + 8 = 31$ balls without selecting a yellow ball. Hence, we must select at least 32 balls to ensure we select at least one of each colour.

(11) Choose any one of the six people, say a. Then take the remaining five people and group those who like a in the set L and those who hate a in the set H. By the CPHP, one of these sets, say L, contains at least three people. If these three people hate each other, then we are done; otherwise, there must be two of them who like each other and, together with a, we then have three people who like each other. Likewise if the set H contains at least three people.

(14) Suppose there is a map from P to H that maps only finitely many elements of P to every element of H. Let m be the maximum number of elements of P mapped to any element of H. Then $|P| \leq m|H|$, which contradicts the fact that P is an infinite set.

(15) There are two possibilities:

Case 1: there are m distinct sets each containing n of the same integers.

$mn + 1 > mn$, so, by the CPHP, with $|P| = mn + 1$, $|H| = m$ and $k = n$, there is one such set containing $n + 1$ equal integers.

Case 2: there are n equal sets containing m distinct integers.

$mn + 1 > mn$, so, by the CPHP (with $|P| = mn + 1$, $|H| = n$ and $k = m$), one of these sets must contain at least $m + 1$ distinct integers.

(16) Note that the size of the subset is not specified, so it makes sense to consider the subsets of $\{m_1, \ldots, m_n\}$ as posets under inclusion. A natural inclusion structure is ascending chains, sequences of sets where one contains the next. If we define the subsets $T_k = \{m_1, \ldots, m_k\}$ for $k = 1, 2, \ldots, n$ the sum corresponding to T_k is $t_k = m_1 + m_2 + \cdots + m_k$ for $1 \leq k \leq n$. If t_k is a multiple of n for some k, we are done. Otherwise, let n be the pigeons and the remainders modulo n the pigeonholes. Then $|P| = n$ and $|H| = n - 1$. Therefore, by the PHP, there are at least two such sums, say t_i and t_j for $i < j$, that have the same remainder after division by n. Hence, by (CON1) of Theorem 5.5.3, n divides $t_j - t_i = m_{i+1} + \cdots + m_j$ and so the set $T_j - T_i$ works.

(17) Consider the sequence of integers

$$m \cdot 1 - 1, \ \ m \cdot 2 - 1, \ldots, \ \ m \cdot n - 1.$$

Choose any two integers in this sequence, say $m \cdot k_1 - 1$ and $m \cdot k_2 - 1 \pmod{n}$. Then $m \cdot k_1 - 1 \equiv m \cdot k_2 - 1 \pmod{n}$ iff $k_1 \equiv k_2$. Thus, no two of the n distinct members of the sequence are congruent modulo n. Therefore, the sequence represents a complete residue system modulo n. However, then one of its members is congruent to 0 modulo n, so there is a positive k such that n divides $m \cdot k - 1$.