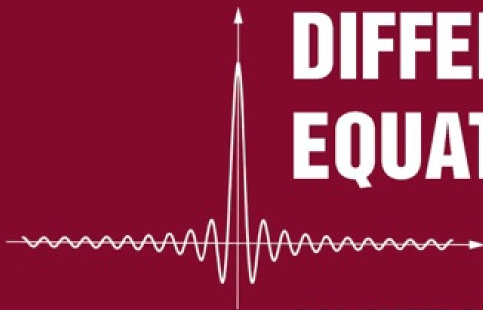
SECOND EDITION

PARTIAL DIFFERENTIAL EQUATIONS



AN INTRODUCTION

Walter A. Strauss

SOLUTIONS MANUAL

Julie L. Levandosky
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Walter A. Strauss

Solutions Manual

for

Partial Differential Equations: An Introduction
Second Edition

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Preface

This is the solutions manual for the Second Edition of the text, Partial Differential Equations: An Introduction" by Walter A. Strauss. We give detailed answers to about half the exercises. The manual is intended to be used by the student as a study guide in conjunction with the text itself. We hope that it will make the text more user-friendly.

The exercises have generally been chosen to be a representative sample. However, they purposely do include many of the more difficult and lengthy ones. In a few cases we have presented more than one method of solution. We have tried to be consistent with the notation of the text, including the numbering of the equations.

A few exercises have changed their numbering from the First Edition. These re-numberings are listed at the end of the manual.

We would appreciate readers pointing out any errors to us.

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Chapter 1

Section 1.1

1.1.2.

(a) $\mathscr{L}$ is linear.

(b) $\mathscr{L}$ is not linear, because of the uu_y term.

1.1.3.

(a) The equation is second-order, linear and inhomogeneous. It takes the form $\mathscr{L}(u) = -1$, where $\mathscr{L}(u) = u_t - u_{xx}$ is linear.

(c) The equation is third-order, nonlinear. The term uu_x is nonlinear.

1.1.5.

(a) The set is a vector space. Given any two vectors $[a_1, 0, c_1]$ and $[a_2, 0, c_2]$ in the set, their sum

$$[a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2]$$

is also in the set, and the scalar multiple

$$c[a_1, 0, c_1] = [ca_1, 0, cc_1]$$

is in the set. Thus the set is a vector space.

(c) The set is not a vector space. Since both $[0, 1, 1]$ and $[1, 0, 3]$ are in the set, but their sum $[1, 1, 4]$ is not, since the product of the first two components is 1, not 0. So the set is not closed under addition. (It is closed under scalar multiplication, but that is irrelevant.)

1.1.7. The functions $1 + x, 1 - x$ and $1 + x + x^2$ are linearly independent. Suppose

$$a(1 + x) + b(1 - x) + c(1 + x + x^2) = 0$$

Then $cx^2 + (a - b + c)x + (a + b + c) = 0$. Since this must hold for every x, all three coefficients c, $a - b + c$ and $a + b + c$ must be zero. Since $c = 0$, the other equations imply $a = b$ and $a = -b$, which implies $a = b = 0$. So the only linear combination of $1 + x$, $1 - x$ and $1 + x + x^2$ which equals zero is the trivial combination.

1.1.10. Let u and v be any two solutions of the differential equation. Then

$$(u + v)''' - 3(u + v)'' + 4(u + v) = u''' - 3u'' + 4u + v''' - 3v'' + 4v = 0 + 0 = 0$$

ghost points are shown in parentheses.

$$\begin{array}{lccccccc}
n=6: & 0 & \frac{35}{8} & \frac{495}{64} & 10 & \frac{735}{64} & \frac{45}{4} & \left(\frac{735}{64}\right) \\[1em]
n=5: & 0 & \frac{75}{16} & \frac{35}{4} & \frac{345}{32} & \frac{45}{4} & \frac{195}{16} & \left(\frac{45}{4}\right) \\[1em]
n=4: & 0 & 6 & \frac{75}{8} & \frac{23}{2} & \frac{195}{16} & 11 & \left(\frac{195}{16}\right) \\[1em]
n=3: & 0 & \frac{51}{8} & 12 & \frac{99}{8} & 11 & 12 & (11) \\[1em]
n=2: & 0 & 10 & \frac{51}{4} & 14 & 12 & 8 & (12) \\[1em]
n=1: & 0 & \frac{21}{2} & 20 & 15 & 8 & 9 & (8) \\[1em]
n=0: & 0 & 24 & 21 & 16 & 9 & 0 & (9)
\end{array}$$

The approximation for $u(3,3)$ is 10.

8.2.8.

(a) The scheme is

$$u_j^{n+1} = u_j^n + \frac{s}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}),$$

or equivalently

$$\frac{s}{2}u_{j+1}^{n+1} - (s+1)u_j^{n+1} + \frac{s}{2}u_{j-1}^{n+1} = -\frac{s}{2}u_{j+1}^n + (s-1)u_j^n - \frac{s}{2}u_{j-1}^n.$$

(b) Let $v(x,t) = u(1-x,t)$. Then $v_t(x,t) = u_t(1-x,t)$, $v_x(x,t) = -u_x(1-x,t)$ and $v_{xx}(x,t) = u_{xx}(1-x,t) = u_t(1-x,t) = v_t(x,t)$. Thus v is a solution of the heat equation. Its initial data is $v(x,0) = u(1-x,0) = \phi(1-x) = \phi(x)$. By uniqueness, it therefore follows that $v(x,t) = u(x,t)$. That is, $u(1-x,t) = u(x,t)$.

(c) First, $s = 6$. So the template is:

$$\begin{array}{ccc}
3/7 & * & 3/7 \\
3/7 & -5/7 & 3/7
\end{array}$$

Next, the boundary condition implies $u_0^1 = u_6^1 = 0$. The scheme in part (a) then implies

$$\begin{array}{rcrcrcl}
3u_2^1 & - & 7u_1^1 & & & = & 0 \\
3u_3^1 & - & 7u_2^1 & + & 3u_1^1 & = & -3 \\
3u_4^1 & - & 7u_3^1 & + & 3u_2^1 & = & 5 \\
3u_5^1 & - & 7u_4^1 & + & 3u_3^1 & = & -3 \\
 & - & 7u_5^1 & + & 3u_4^1 & = & 0.
\end{array}$$

Solving this system of simultaneous linear equations gives

$$0 \quad \frac{9}{77} \quad \frac{3}{11} \quad -\frac{37}{77} \quad \frac{3}{11} \quad \frac{9}{77} \quad 0$$

after one time step.

equation is $R(\rho) = C_n J_n(\rho) + D_n N_n(\rho)$. Hence

$$u(r,\theta) = (C_n J_n(\sqrt{\lambda}r) + D_n N_n(\sqrt{\lambda}r))(A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition $u(a,\theta) = 0$ implies $D_n = -\frac{J_n(\sqrt{\lambda}a)}{N_n(\sqrt{\lambda}a)}C_n$, so after absorbing the remaining constant into A_n and B_n, we have

$$u(r,\theta) = (N_n(\sqrt{\lambda}a)J_n(\sqrt{\lambda}r) - J_n(\sqrt{\lambda}a)N_n(\sqrt{\lambda}r))(A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition $u(b,\theta) = 0$ then implies that λ must be a root of

$$N_n(\sqrt{\lambda}a)J_n(\sqrt{\lambda}b) - J_n(\sqrt{\lambda}a)N_n(\sqrt{\lambda}b) = 0.$$

Section 10.6

10.6.5. For any solution u of Legendre's equation, integration by parts gives

$$\begin{aligned}
\gamma \int_0^1 xu(x)\,dx &= -\int_0^1 x[(1-x^2)u'(x)]'\,dx \\
&= -x(1-x^2)u'(x)\Big|_0^1 + \int_0^1 (1-x^2)u'(x)\,dx \\
&= u(x)(1-x^2)\Big|_0^1 + 2\int_0^1 xu(x)\,dx,
\end{aligned}$$

so

$$\int_0^1 xu(x)\,dx = -\frac{u(0)}{\gamma - 2}$$

for $\gamma \neq 2$. Applying this with $\gamma = l(l+1)$ and $u = P_l$ gives

$$\int_{-1}^1 f(x)P_l(x)\,dx = \int_0^1 xP_l(x)\,dx = -\frac{P_l(0)}{(l-1)(l+2)},$$

for $l \neq 1$. By equation (10.6.3), $P_l(0) = 0$ for all odd l and

$$P_{2n}(0) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}.$$

Hence,

$$\int_{-1}^1 f(x)P_{2n}(x)\,dx = \frac{(-1)^{n+1}(2n-2)!}{2^{2n}(n+1)!(n-1)!}$$

for $n \geq 1$. Combining this with equation (10.6.6) gives

$$a_{2n} = \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2\cdot 4^n(n+1)!(n-1)!}$$

Section 13.4

13.4.2. The equation is

$$-\frac{d^2\psi}{dx^2} - Q\delta(x)\psi(x) = \lambda\psi(x).$$

If λ is an eigenvalue with an eigenfunction $\psi(x)$, then by definition $\int |\psi|^2 dx < \infty$. So we are looking for solutions $\not\equiv 0$ that decay at infinity. For $x \neq 0$, the potential term is completely missing. So the solutions are exponentials if $\lambda < 0$, while the solutions do not decay if $\lambda \geq 0$ (because they are sines and cosines for $\lambda > 0$). So we can write $\lambda = -\beta^2 < 0$ and conclude that

$$\psi(x) = Ce^{-\beta|x|} \quad \text{for } x \neq 0.$$

Differentiating, we get

$$\psi'(x) = -C\beta e^{-\beta|x|}\text{sign}(x)$$

and

$$\psi''(x) = -C\beta e^{-\beta|x|}2\delta(x) + C\beta^2 e^{-\beta|x|}.$$

See (12.1.15) for an explanation of the last differentiation. Now insert these expressions into the ODE to get

$$0 = -\frac{d^2\psi}{dx^2} - Q\delta(x)\psi(x) - \beta^2\psi(x) = Ce^{-\beta|x|}\{-2\beta\delta(x) + \beta^2 + Q\delta(x) - \beta^2\}.$$

This is true if and only if $Q = 2\beta$. Thus the only eigenvalue and eigenfunction are

$$\lambda = -\frac{Q^2}{4} < 0\,, \quad \psi(x) = Ce^{-\sqrt{-\lambda}\ |x|}.$$

(The rest of the spectrum is continuous.)

Section 13.5

13.5.3.

(a) Proof that $\mathscr{E}$ is an invariant:

$$\frac{\partial\mathscr{E}}{\partial t} = \iiint \left(\mathbf{E}_1 \cdot \frac{\partial \mathbf{E}_1}{\partial t} + \cdots + \mathbf{B}_3 \cdot \frac{\partial \mathbf{B}_3}{\partial t}\right) d\mathbf{x}.$$

The first term in this integrand is

$$\begin{aligned}\mathbf{E}_1 \cdot \frac{\partial \mathbf{E}_1}{\partial t} &= \mathbf{E}_1 \cdot (D_0\mathbf{E}_1 + \mathbf{A}_0 \times \mathbf{E}_1) = \mathbf{E}_1 \cdot D_0\mathbf{E}_1 \\ &= \mathbf{E}_1 \cdot \left(\frac{\partial \mathbf{B}_3}{\partial x_2} + \mathbf{A}_2 \times \mathbf{B}_3 - \frac{\partial \mathbf{B}_2}{\partial x_3} - \mathbf{A}_3 \times \mathbf{B}_2\right).\end{aligned}$$