SOLUTIONS TO CHAPTER 2
Background

2.1 The DFT of a sequence $x(n)$ of length N may be expressed in matrix form as follows

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

where $\mathbf{x} = [x(0),\ x(1),\ \ldots, x(N-1)]^T$ is a vector containing the signal values and $\mathbf{X}$ is a vector containing the DFT coefficients $X(k)$,

(a) Find the matrix $\mathbf{W}$.

(b) What properties does the matrix $\mathbf{W}$ have?

(c) What is the inverse of $\mathbf{W}$?

Solution

(a) The DFT of a sequence $x(n)$ of length N is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

where $W_N \equiv e^{-j\frac{2\pi}{N}}$. If we define

$$\mathbf{w}_k^H = \left[1, W_N^k, W_N^{2k}, \ldots, W_N^{k(N-1)}\right]$$

then $X(k)$ is the inner product

$$X(k) = \mathbf{w}_k^H \cdot \mathbf{x}$$

Arranging the DFT coefficients in a vector we have,

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{w}_0^H\mathbf{x} \\ \mathbf{w}_1^H\mathbf{x} \\ \vdots \\ \mathbf{w}_{N-1}^H\mathbf{x} \end{bmatrix} = \mathbf{W}\mathbf{x}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0^H \\ \mathbf{w}_1^H \\ \vdots \\ \mathbf{w}_{N-1}^H \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

(b) The matrix $\mathbf{W}$ is *symmetric* and nonsingular. In addition, due to the orthogonality of the complex exponentials,

$$\mathbf{w}_k^H \cdot \mathbf{w}_l = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)} = \begin{cases} N & ; \quad k = l \\ 0 & ; \quad k \neq l \end{cases}$$

it follows that $\mathbf{W}$ is *orthogonal.*

(c) Due to the orthogonality of $\mathbf{W}$, the inverse is

$$\mathbf{W}^{-1} = \tfrac{1}{N}\mathbf{W}^H$$

2.7 Show that a projection matrix $\mathbf{P}_A$ has the following two properties,

1. It is *idempotent*, $\mathbf{P}_A^2 = \mathbf{P}_A$.
2. It is Hermitian.

Solution

Given a matrix $\mathbf{A}$, the projection matrix $\mathbf{P}_A$ is

$$\mathbf{P}_A = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

Therefore,

$$\begin{aligned}\mathbf{P}_A^2 &= \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H \\ &= \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H = \mathbf{P}_A\end{aligned}$$

and it follows that $\mathbf{P}_A$ is idempotent. Also,

$$\mathbf{P}_A^H = \left[\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\right]^H = \mathbf{A}\left[(\mathbf{A}^H\mathbf{A})^{-1}\right]^H\mathbf{A}^H$$

Since $\mathbf{A}\mathbf{A}^H$ is Hermitian, then so is its inverse,

$$\left[(\mathbf{A}^H\mathbf{A})^{-1}\right]^H = (\mathbf{A}^H\mathbf{A})^{-1}$$

and

$$\mathbf{P}_A^H = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

Thus, $\mathbf{P}_A$ is Hermitian.

then the normal equations become (recall that $a_p(0) = 1$)

$$\sum_{l=1}^{p} a_p(l) r_x(k,l) - b(0)x(-k) - b(0)x(n_0 - k) = -r_x(k,0) \quad ; \quad k = 1, 2, \ldots, p$$

Assuming that $x(n) = 0$ for $n < 0$, with $\mathbf{x} = [x(n_0-1), x(n_0-2), \ldots, x(n_0-p)]^T$, the normal equations may be written in matrix form as follows

$$\mathbf{R}_x\mathbf{a} - b(0)\mathbf{x} = -\mathbf{r}_x$$

Finally, differentiating with respect to $b(0)$ we have

$$\frac{\partial \mathcal{E}}{\partial b(0)} = -\sum_{n=0}^{\infty} 2 \left[\sum_{l=0}^{p} a_p(l)x(n-l) - b(0)\delta(n) - b(0)\delta(n-n_0) \right] [\delta(n) + \delta(n-n_0)]$$

Thus,

$$x(0) - b(0) + \sum_{l=1}^{p} a_p(l)x(n_0 - l) - b(0) = -x(n_0)$$

or, in vector form, we have

$$\mathbf{x}^T\mathbf{a} - 2b(0) = -x(0) - x(n_0)$$

Putting all of these together in matrix form yields

$$\begin{bmatrix} \mathbf{R}_x & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ -2b(0) \end{bmatrix} = -\begin{bmatrix} \mathbf{r}_x \\ x(0) + x(n_0) \end{bmatrix}$$

7.15 Let $x(n)$ be an AR(1) process of the following form

$$x(n) = a(1)x(n-1) + b(0)w(n)$$

where $w(n)$ is unit variance white noise, and let $y(n)$ be noisy measurements

$$y(n) = x(n) + v(n)$$

where $v(n)$ is unit variance white noise that is uncorrelated with $w(n)$. We have seen that the causal Wiener filter for estimating $x(n)$ from $y(n)$ has the form

$$\widehat{x}(n) = a(1)\widehat{x}(n-1) + K\Big[y(n) - a(1)\widehat{x}(n-1)\Big]$$

Find the value of K in terms of $a(1)$ and $b(0)$ that minimizes the mean-square error

$$E\{[x(n) - \widehat{x}(n)]^2\}$$

Solution

With an estimate of the form

$$\widehat{x}(n) = a(1)\widehat{x}(n-1) + K\Big[y(n) - a(1)\widehat{x}(n-1)\Big]$$

we want to find the value of K that minimizes the mean-square error

$$\xi = E\Big\{\big[x(n) - \widehat{x}(n)\big]^2\Big\}$$

This problem may be solved by differentiating ξ with respect to K, and set the result equal to zero. After a fair amount of work, we find that

$$K = \frac{b^2(0) + a^2(1)\xi_{\min}}{1 + b^2(0) + a^2(1)\xi_{\min}}$$

Unfortunately, however, $\xi_{\min}$ depends upon K. Using the expression

$$\xi_{\min} = r_x(0) - \sum_{l=0}^{\infty} h(l)r_{xy}^*(l)$$

with

$$h(n) = K\Big(\big[1-K\big]a(1)\Big)^n u(n)$$

and

$$r_{xy}(k) = r_x(k) = \frac{b^2(0)}{1-a^2(1)}a(1)^{|k|}$$

we may easily derive the following expression for $\xi_{\min}$

$$\xi_{\min} = b^2(0)\frac{1-K}{1-(1-K)a^2(1)}$$

Solving these two equations for K leads to the following quadratic equation,

$$a^2(1)K^2 + [1 + b^2(0) - a^2(1)]K - b^2(0) = 0$$

and the desired solution is the positive real root of this quadratic. Note that if we substitute $a(1) = 0.8$ and $b(0) = 0.6$ we arrive at the values for K and $\xi_{\min}$ derived in Example 7.3.2.

9.19 Adaptive filters are commonly used for linear prediction. Although harmonic signals such as sinusoids are perfectly predictable, measurement noise will degrade the performance of the predictor and add a bias to the coefficients, $\mathbf{w}$. For example, suppose that we want to design an adaptive linear predictor for a real-valued process $x(n)$ using the noisy measurements

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero mean white noise that is uncorrelated with $x(n)$. Assume that the variance of $v(n)$ is σ_v^2.

(a) Using the LMS algorithm

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e(n)\mathbf{y}(n)$$

find the range of values for μ for which the LMS algorithm converges in the mean and find

$$\lim_{n\to\infty} E\{\mathbf{w}_n\}$$

(b) The γ-LMS algorithm has been proposed as an adaptive filtering algorithm to combat the effect of measurement noise. Using the noisy observations, $y(n)$, this algorithm is

$$\mathbf{w}_{n+1} = \gamma\mathbf{w}_n + \mu e(n)\mathbf{y}(n)$$

where γ is a constant. Explain how the γ-LMS algorithm can be used to remove the bias in the steady-state solution of the LMS algorithm. Specifically, how would you select values for μ and γ?

Solution

(a) The LMS algorithm is

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e(n)\mathbf{y}(n)$$

Taking expected values we find

$$E\{\mathbf{w}_{n+1}\} = \left[\mathbf{I} - \mu\mathbf{R}_y\right]E\{\mathbf{w}_n\} + \mu\mathbf{r}_{dy}$$

where

$$\mathbf{R}_y = \mathbf{R}_x + \sigma_v^2\mathbf{I}$$

is the autocorrelation matrix of $\mathbf{y}(n)$. Since the eigenvalues of $\mathbf{R}_y$ are

$$\tilde{\lambda}_k = \lambda_k + \sigma_v^2$$

where λ_k are the eigenvalues of $\mathbf{R}_x$, then the LMS algorithm converges in the mean if

$$0 < \mu < \frac{2}{\lambda_{\max} + \sigma_v^2}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{R}_x$. Furthermore, if the LMS algorithm converges in the mean, then

$$\lim_{n\to\infty} E\{\mathbf{w}_n\} = (\mathbf{R}_x + \sigma_v^2\mathbf{I})^{-1}\mathbf{r}_{dx}$$