SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

(1) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d \ln Z(t)}{dt} = \frac{d \ln[X(t)Y(t)]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

(2) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) + \ln Y(t)]}{dt} = \frac{d \ln X(t)}{dt} + \frac{d \ln Y(t)}{dt},$$

or simply

(3) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

(4) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d \ln Z(t)}{dt} = \frac{d \ln[X(t)/Y(t)]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

(5) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) - \ln Y(t)]}{dt} = \frac{d \ln X(t)}{dt} - \frac{d \ln Y(t)}{dt},$$

or simply

(6) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

(7) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d \ln Z(t)}{dt} = \frac{d \ln[X(t)^{\alpha}]}{dt}.$$

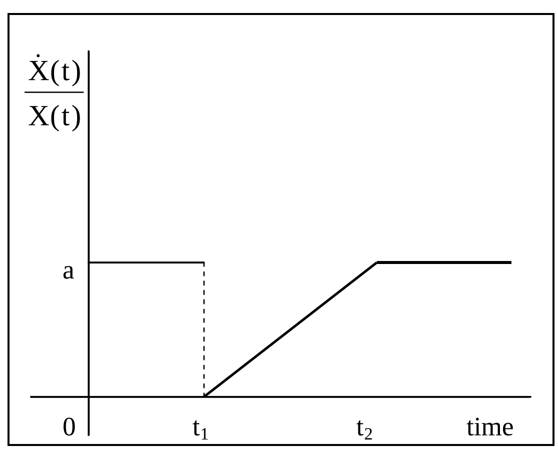
Using the fact that $\ln[X(t)^{\alpha}] = \alpha \ln X(t)$, we have

(8) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\alpha \ln X(t)]}{dt} = \alpha \frac{d \ln X(t)}{dt} = \alpha \frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



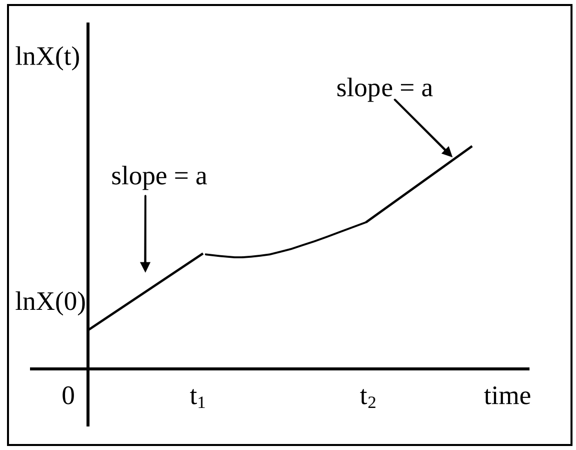
From time 0 to time t_1, the growth rate of X is constant and equal to a > 0. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

(b) Note that the slope of lnX(t) plotted against time is equal to the growth rate of X(t). That is, we know

$$\frac{d\ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of lnX(t) equals a > 0. The lnX(t) locus has an inflection point at t_1, when the growth rate of X(t) changes discontinuously from a to 0. Between t_1 and t_2, the slope of lnX(t) rises gradually from 0 to a. After time t_2 the slope of lnX(t) is constant and equal to a > 0 again.

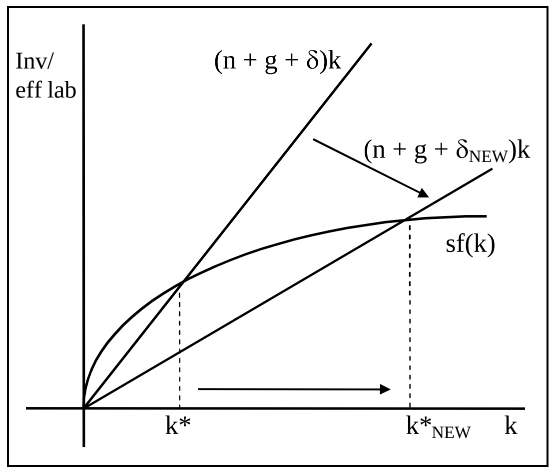


Problem 1.3

(a) The slope of the break-even investment line is given by (n + g + δ) and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

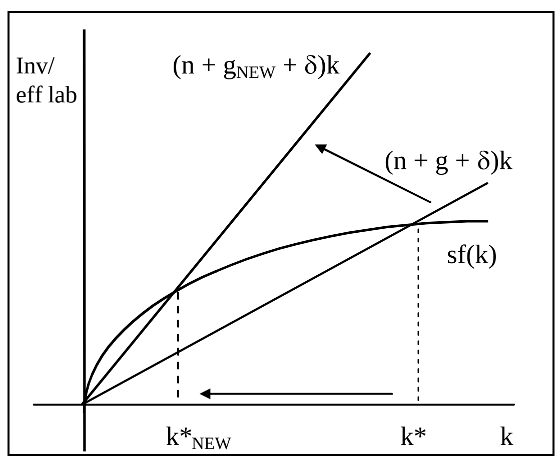
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW} .



(b) Since the slope of the break-even investment line is given by (n + g + δ), a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, sf(k), is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k* to k^*_{NEW} .



The individual can sell one unit of the good in period t for Q_t. In period t + 1, it costs Q_{t+1} to obtain one unit of the good or equivalently, it costs one to obtain $1/Q_{t+1}$ units of the good. Thus for Q_t, it is possible to obtain Q_t/Q_{t+1} units of the good. Thus selling a unit of the good in period t allows the individual to buy Q_t/Q_{t+1} units of the good in period t + 1. Thus the gross rate of return on trading is Q_t/Q_{t+1}.

Now, $Q_{t+1} = Q_t/x$ for all t > 0 is equivalent to $x = Q_t/Q_{t+1}$ for all t > 0. In other words, the rate of return on storage is equal to the rate of return on trading and hence the individual is indifferent as to the amount to store and the amount to trade. Let $\alpha_t \in [0, 1]$ represent the fraction of "saving", A/2, that the individual sells in period t. That is, the individual sells $\alpha_t(A/2)$ in period t. This allows the individual to buy the amount $\alpha_t(Q_t/Q_{t+1})(A/2)$ when she is old in period t + 1. The individual stores a fraction $(1 - \alpha_t)$ of her "saving". Thus

(3) $S_t = (1 - \alpha_t)(A/2)$.

Consumption in period t + 1 will be equal to the amount the individual buys plus the amount she has through storage. Thus

(4) $C_{2,t+1} = \alpha_t(Q_t/Q_{t+1})(A/2) + (1 - \alpha_t)x(A/2)$.

Since we are considering a case in which $Q_t/Q_{t+1} = x$, equation (4) can be rewritten as

(5) $C_{2,t+1} = \alpha_t x(A/2) + (1 - \alpha_t)x(A/2) = x(A/2)$.

Consider some period t + 1 and let L represent the total number of individuals born each period, which is constant. Aggregate supply in period t + 1 is equal to the total number of young individuals, L, multiplied by the amount that each young individual wishes to sell, $\alpha_{t+1}(A/2)$. Thus

(6) Aggregate $\text{Supply}_{t+1} = L\alpha_{t+1}(A/2)$.

Aggregate demand in period t + 1 is equal to the total number of old individuals, L, multiplied by the amount each old individual wishes to buy, $(Q_t/Q_{t+1})\alpha_t(A/2)$. Thus

(7) Aggregate $\text{Demand}_{t+1} = L(Q_t/Q_{t+1})\alpha_t(A/2)$.

For the market to clear, aggregate supply must equal aggregate demand or

(8) $L\alpha_{t+1}(A/2) = L(Q_t/Q_{t+1})\alpha_t(A/2)$,

or simply

(9) $\alpha_{t+1} = (Q_t/Q_{t+1})\alpha_t$.

Since the proposed price path has $Q_{t+1} = Q_t/x$, the equilibrium condition given by equation (9) can also be written as

(10) $\alpha_{t+1} = x\alpha_t$.

Now consider the situation in period 0. The old individuals simply consume their endowment. Thus we must have α_0 equal to zero in order for the market to clear in period 0. Thus equation (10) implies that we must have $\alpha_t = 0$ for all $t \geq 0$.

The resulting equilibrium is the same as that in part (a) of Problem 2.18. The individual consumes half of her endowment in the first period of life, stores the rest and consumes xA/2 in the second period of life. Note that with x < 1 + n here (since n = 0 and x < 1), this equilibrium is dynamically inefficient. Thus eliminating incomplete markets by allowing individuals to trade before the start of time does not eliminate dynamic inefficiency.

(a) (ii) Suppose the auctioneer announces $Q_{t+1} < Q_t/x$ or equivalently $x < (Q_t/Q_{t+1})$ for some date t. This means that trading dominates storage for the young at date t. This means that the young at date t will want to sell all of their saving – $\alpha_t = 1$ so that they want to sell A/2 – and not store anything. Thus aggregate supply in period t is equal to L(A/2). For the old at date t, Q_{t+1} is irrelevant. They based their decision of how much to buy when old on Q_t/Q_{t-1} which was equal to x. Thus as described in part (a) (i), old individuals were not planning to buy anything. Thus aggregate demand in period t is zero. Thus aggregate demand will be less than aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

(7) $C_2 = Y_2 - (1 + E[r] + \varepsilon)B_1 = Y_2 - (1 + E[r] + \varepsilon)C_1$,

where B_1 represents the amount of borrowing the individual does in the first period. Substituting (7) into the expected utility function (1) yields

(8) $U = \ln C_1 + E[\ln(Y_2 - (1 + E[r] + \varepsilon)C_1)]$.

Set the derivative of equation (8) with respect to C_1 equal to zero to find the first-order condition:

(9) $\partial U/\partial C_1 = 1/C_1 - E\left[(1 + E[r] + \varepsilon)/C_2\right] = 0$.

Use the formula for the expected value of the product of 2 random variables – E[XY] = E[X]E[Y] + cov(X,Y) – to obtain

(10) $1/C_1 = (1 + E[r])E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right)$.

The covariance term is positive. Intuitively, a higher ε means the individual has to pay more interest on her borrowing which forces her to have lower C_2 and thus higher $1/C_2$.

If r is not random – so that r = E[r] with certainty – we can rewrite equation (10) as

(11) $1/C_1 = (1 + E[r])(1/C_2) = (1 + E[r])/\left[Y_2 - (1 + E[r])C_1\right]$,

which implies that

(12) $Y_2 - (1 + E[r])C_1 = (1 + E[r])C_1$.

Solving for C_1 yields

(13) $C_1 = Y_2/2(1 + E[r])$.

In the case where r is random, equation (10) can be written as

(14) $1/C_1 = E[1 + E[r] + \varepsilon]\,E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right)$.

Since $1/C_2$ is a convex function of C_2 , then by Jensen's inequality we have $E[1/C_2] > 1/E[C_2]$. In addition, because the covariance term is positive, we can write

(15) $1/C_1 = (1 + E[r])E[1/C_2] + \text{cov}\left(1 + E[r] + \varepsilon, 1/C_2\right) > (1 + E[r])\left[1/E[C_2]\right]$.

Substituting into this inequality the fact that $E[C_2] = Y_2 - (1 + E[r])C_1$ yields

(16) $1/C_1 > (1 + E[r])/\left[Y_2 - (1 + E[r])C_1\right]$,

which implies that

(17) $Y_2 - (1 + E[r])C_1 > (1 + E[r])C_1$.

The inequality in (17) can be rewritten as

(18) $2(1 + E[r])C_1 < Y_2$

or simply

(19) $C_1 < Y_2/2(1 + E[r])$.

Note from equation (13) that the right-hand side of (19) is the optimal choice of C_1 under certainty. Thus we have shown that if r becomes random with no change in the expected value of r, the optimal choice of C_1 becomes smaller. Essentially, if there is some uncertainty about how much interest the individual will have to pay in the second period, she is more cautious in her decision as to how much to borrow and consume in the first period.

Problem 5.7

(a) Imagine the household increasing its labor supply per member in period t by a small amount Δl. Suppose it then uses the resulting greater wealth to allow less labor supply per member in the next period and allowing for consumption per member to be the same in both periods as it otherwise would have been. If the household is behaving optimally, a marginal change of this type must leave expected lifetime utility unchanged.

Household utility and the instantaneous utility function of the representative member of the household are given by

Problem 8.6

(a) Substituting the expression for consumption in period t, which is

$$(1)\ \ C_t = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s}\right],$$

into the expression for wealth in period t + 1, which is

(2) $A_{t+1} = (1 + r)[A_t + Y_t - C_t]$,

gives us

$$(3)\ \ A_{t+1} = (1+r)\left[A_t + Y_t - \frac{r}{1+r}A_t - \frac{r}{1+r}\left(Y_t + \frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right)\right].$$

Obtaining a common denominator of (1 + r) and then canceling the (1 + r)'s gives us

$$(4)\ \ A_{t+1} = A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right).$$

Since equation (1) holds in all periods, we can write consumption in period t + 1 as

$$(5)\ \ C_{t+1} = \frac{r}{1+r}\left[A_{t+1} + \sum_{s=0}^{\infty}\frac{E_{t+1}[Y_{t+1+s}]}{(1+r)^s}\right].$$

Substituting equation (4) into equation (5) yields

$$(6)\ \ C_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_{t+1}Y_{t+1} + \frac{E_{t+1}Y_{t+2}}{1+r} + \ldots\right)\right].$$

Taking the expectation, conditional on time t information, of both sides of equation (6) gives us

$$(7)\ \ E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - r\left(\frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right) + \left(E_tY_{t+1} + \frac{E_tY_{t+2}}{1+r} + \ldots\right)\right],$$

where we have used the law of iterated projections so that for any variable x, $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. If this did not hold, individuals would be expecting to revise their estimate either upward or downward and thus their original expectation could not have been rational. Collecting terms in equation (7) gives us

$$(8)\ \ E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t - \left(1 - \frac{r}{1+r}\right)E_tY_{t+1} + \left(\frac{1}{1+r} - \frac{r}{(1+r)^2}\right)E_tY_{t+2} + \ldots\right],$$

which simplifies to

$$(9)\ \ E_tC_{t+1} = \frac{r}{1+r}\left[A_t + Y_t + \frac{E_tY_{t+1}}{1+r} + \frac{E_tY_{t+2}}{(1+r)^2} + \ldots\right].$$

Using summation notation, and noting that $E_t Y_t = Y_t$, we have

$$(10)\ \ E_tC_{t+1} = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_tY_{t+s}}{(1+r)^s}\right].$$

The right-hand sides of equations (1) and (10) are equal and thus

(11) $E_t C_{t+1} = C_t$.

Consumption follows a random walk; changes in consumption are unpredictable.

Since consumption follows a random walk, the best estimate of consumption in any future period is simply the value of consumption in this period. That is, for any $s \geq 0$, we can write

(12) $E_t C_{t+s} = C_t$.

Using equation (12), we can write the present value of the expected path of consumption as