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One Thousand Exercises in Probability

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Preface

This book contains more than 1000 exercises in probability and random processes, together with their solutions. Apart from being a volume of worked exercises in its own right, it is also a solutions manual for exercises and problems appearing in our textbook *Probability and Random Processes* (3rd edn), Oxford University Press, 2001, henceforth referred to as PRP. These exercises are not merely for drill, but complement and illustrate the text of PRP, or are entertaining, or both. The current volume extends our earlier book *Probability and Random Processes: Problems and Solutions*, and includes in addition around 400 new problems. Since many exercises have multiple parts, the total number of interrogatives exceeds 3000.

Despite being intended in part as a companion to PRP, the present volume is as self-contained as reasonably possible. Where knowledge of a substantial chunk of bookwork is unavoidable, the reader is provided with a reference to the relevant passage in PRP. Expressions such as 'clearly' appear frequently in the solutions. Although we do not use such terms in their Laplacian sense to mean 'with difficulty', to call something 'clear' is not to imply that explicit verification is necessarily free of tedium.

The table of contents reproduces that of PRP; the section and exercise numbers correspond to those of PRP; there are occasional references to examples and equations in PRP. The covered range of topics is broad, beginning with the elementary theory of probability and random variables, and continuing, via chapters on Markov chains and convergence, to extensive sections devoted to stationarity and ergodic theory, renewals, queues, martingales, and diffusions, including an introduction to the pricing of options. Generally speaking, *exercises* are questions which test knowledge of particular pieces of theory, while *problems* are less specific in their requirements. There are questions of all standards, the great majority being elementary or of intermediate difficulty. We ourselves have found some of the later ones to be rather tricky, but have refrained from magnifying any difficulty by adding asterisks or equivalent devices. If you are using this book for self-study, our advice would be not to attempt more than a respectable fraction of these at a first read.

We pay tribute to all those anonymous pedagogues whose examination papers, work assignments, and textbooks have been so influential in the shaping of this collection. To them and to their successors we wish, in turn, much happy plundering. If you find errors, try to keep them secret, except from us. If you know a better solution to any exercise, we will be happy to substitute it in a later edition.

We acknowledge the expertise of Sarah Shea-Simonds in preparing the TEXscript of this volume, and of Andy Burbanks in advising on the front cover design, which depicts a favourite confluence of the authors.

Cambridge and Oxford G. R. G.
April 2001 D. R. S.

Life is good for only two things, discovering mathematics and teaching it.

Siméon Poisson

In mathematics you don't understand things, you just get used to them.

John von Neumann

Probability is the bane of the age.

Anthony Powell
Casanova's Chinese Restaurant

The traditional professor writes a, says b, and means c; but it should be d.

George Pólya

Contents

1

Events and their probabilities

1.2 Exercises. Events as sets

1. Let $\{A_i : i \in I\}$ be a collection of sets. Prove 'De Morgan's Laws'†:

$$\Big(\bigcup_i A_i\Big)^c = \bigcap_i A_i^c, \qquad \Big(\bigcap_i A_i\Big)^c = \bigcup_i A_i^c.$$

2. Let A and B belong to some σ-field $\mathcal{F}$. Show that $\mathcal{F}$ contains the sets $A \cap B$, $A \setminus B$, and $A \triangle B$.

3. A conventional knock-out tournament (such as that at Wimbledon) begins with 2^n competitors and has n rounds. There are no play-offs for the positions 2, 3, ..., $2^n - 1$, and the initial table of draws is specified. Give a concise description of the sample space of all possible outcomes.

4. Let $\mathcal{F}$ be a σ-field of subsets of Ω and suppose that $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field of subsets of B.

5. Which of the following are identically true? For those that are not, say when they are true.
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
(b) $A \cap (B \cap C) = (A \cap B) \cap C$;
(c) $(A \cup B) \cap C = A \cup (B \cap C)$;
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

1.3 Exercises. Probability

1. Let A and B be events with probabilities $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \le \mathbb{P}(A \cap B) \le \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $\mathbb{P}(A \cup B)$.

2. A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later. Show similarly that any given finite sequence of heads and tails occurs eventually with probability one. Explain the connection with Murphy's Law.

3. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

† Augustus De Morgan is well known for having given the first clear statement of the principle of mathematical induction. He applauded probability theory with the words: "The tendency of our study is to substitute the satisfaction of mental exercise for the pernicious enjoyment of an immoral stimulus".

4. Let $A_1, A_2, \dots, A_n$ be events where $n \ge 2$, and prove that

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n).$$

In each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any given packet contains any specific Vice-Chancellor being $\frac{1}{5}$, independently of all other packets. Show that the probability that each of the last three Vice-Chancellors is obtained in a bulk purchase of six packets is $1 - 3(\frac{4}{5})^6 + 3(\frac{3}{5})^6 - (\frac{2}{5})^6$.

5. Let A_r, $r \ge 1$, be events such that $\mathbb{P}(A_r) = 1$ for all r. Show that $\mathbb{P}\big(\bigcap_{r=1}^{\infty} A_r\big) = 1$.

6. You are given that at least one of the events A_r, $1 \le r \le n$, is certain to occur, but certainly no more than two occur. If $\mathbb{P}(A_r) = p$, and $\mathbb{P}(A_r \cap A_s) = q$, $r \ne s$, show that $p \ge 1/n$ and $q \le 2/n$.

7. You are given that at least one, but no more than three, of the events A_r, $1 \le r \le n$, occur, where $n \ge 3$. The probability of at least two occurring is $\frac{1}{2}$. If $\mathbb{P}(A_r) = p$, $\mathbb{P}(A_r \cap A_s) = q$, $r \ne s$, and $\mathbb{P}(A_r \cap A_s \cap A_t) = x$, $r < s < t$, show that $p \ge 3/(2n)$, and $q \le 4/n$.

1.4 Exercises. Conditional probability

1. Prove that $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\mathbb{P}(A)/\mathbb{P}(B)$ whenever $\mathbb{P}(A)\mathbb{P}(B) \ne 0$. Show that, if $\mathbb{P}(A \mid B) > \mathbb{P}(A)$, then $\mathbb{P}(B \mid A) > \mathbb{P}(B)$.

2. For events $A_1, A_2, \dots, A_n$ satisfying $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, prove that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

3. A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

4. What do you think of the following 'proof' by Lewis Carroll that an urn cannot contain two balls of the same colour? Suppose that the urn contains two balls, each of which is either black or white; thus, in the obvious notation, $\mathbb{P}(\mathrm{BB}) = \mathbb{P}(\mathrm{BW}) = \mathbb{P}(\mathrm{WB}) = \mathbb{P}(\mathrm{WW}) = \frac{1}{4}$. We add a black ball, so that $\mathbb{P}(\mathrm{BBB}) = \mathbb{P}(\mathrm{BBW}) = \mathbb{P}(\mathrm{BWB}) = \mathbb{P}(\mathrm{BWW}) = \frac{1}{4}$. Next we pick a ball at random; the chance that the ball is black is (using conditional probabilities) $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{2}{3}$. However, if there is probability $\frac{2}{3}$ that a ball, chosen randomly from three, is black, then there must be two black and one white, which is to say that originally there was one black and one white ball in the urn.

5. **The Monty Hall problem: goats and cars.** (a) Cruel fate has made you a contestant in a game show; you have to choose one of three doors. One conceals a new car, two conceal old goats. You

(b) Find the joint density function of $X + Y$ and $X/(X + Y)$, and deduce that they are independent.

(c) If Z is Poisson with parameter λt, and m is integral, show that $\mathbb{P}(Z < m) = \mathbb{P}(X > t)$.

(d) If $0 < m < n$ and B is independent of Y with the beta distribution with parameters m and $n - m$, show that YB has the same distribution as X.

12. Let $X_1, X_2, \ldots, X_n$ be independent $N(0, 1)$ variables.

(a) Show that X_1^2 is $\chi^2(1)$.

(b) Show that $X_1^2 + X_2^2$ is $\chi^2(2)$ by expressing its distribution function as an integral and changing to polar coordinates.

(c) More generally, show that $X_1^2 + X_2^2 + \cdots + X_n^2$ is $\chi^2(n)$.

13. Let X and Y have the bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation ρ. Show that

(a) $\mathbb{E}(X \mid Y) = \mu_1 + \rho\sigma_1(Y - \mu_2)/\sigma_2$,

(b) the variance of the conditional density function $f_{X|Y}$ is $\mathrm{var}(X \mid Y) = \sigma_1^2(1 - \rho^2)$.

14. Let X and Y have joint density function f. Find the density function of Y/X.

15. Let X and Y be independent variables with common density function f. Show that $\tan^{-1}(Y/X)$ has the uniform distribution on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ if and only if

$$\int_{-\infty}^{\infty} f(x)f(xy)|x|\,dx = \frac{1}{\pi(1 + y^2)}, \qquad y \in \mathbb{R}.$$

Verify that this is valid if either f is the $N(0, 1)$ density function or $f(x) = a(1 + x^4)^{-1}$ for some constant a.

16. Let X and Y be independent $N(0, 1)$ variables, and think of (X, Y) as a random point in the plane. Change to polar coordinates (R, Θ) given by $R^2 = X^2 + Y^2$, $\tan\Theta = Y/X$; show that R^2 is $\chi^2(2)$, $\tan\Theta$ has the Cauchy distribution, and R and Θ are independent. Find the density of R.

Find $\mathbb{E}(X^2/R^2)$ and

$$\mathbb{E}\left\{\frac{\min\{|X|, |Y|\}}{\max\{|X|, |Y|\}}\right\}.$$

17. If X and Y are independent random variables, show that $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$ have distribution functions

$$F_U(u) = 1 - \{1 - F_X(u)\}\{1 - F_Y(u)\}, \qquad F_V(v) = F_X(v)F_Y(v).$$

Let X and Y be independent exponential variables, parameter 1. Show that

(a) U is exponential, parameter 2,

(b) V has the same distribution as $X + \frac{1}{2}Y$. Hence find the mean and variance of V.

18. Let X and Y be independent variables having the exponential distribution with parameters λ and μ respectively. Let $U = \min\{X, Y\}$, $V = \max\{X, Y\}$, and $W = V - U$.

(a) Find $\mathbb{P}(U = X) = \mathbb{P}(X \le Y)$.

(b) Show that U and W are independent.

19. Let X and Y be independent non-negative random variables with continuous density functions on $(0, \infty)$.

(a) If, given $X + Y = u$, X is uniformly distributed on $[0, u]$ whatever the value of u, show that X and Y have the exponential distribution.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let H be the space of $\mathcal{G}$-measurable random variables with finite second moment.

(a) Show that H is closed with respect to the norm $\|\cdot\|_2$.

(b) Let Y be a random variable satisfying $\mathbb{E}(Y^2) < \infty$, and show the equivalence of the following two statements for any $M \in H$:

(i) $\mathbb{E}\{(Y - M)Z\} = 0$ for all $Z \in H$,

(ii) $\mathbb{E}\{(Y - M)I_G\} = 0$ for all $G \in \mathcal{G}$.

7.10 Exercises. Uniform integrability

1. Show that the sum $\{X_n + Y_n\}$ of two uniformly integrable sequences $\{X_n\}$ and $\{Y_n\}$ gives a uniformly integrable sequence.

2. (a) Suppose that $X_n \xrightarrow{r} X$ where $r \geq 1$. Show that $\{|X_n|^r : n \geq 1\}$ is uniformly integrable, and deduce that $\mathbb{E}(X_n^r) \to \mathbb{E}(X^r)$ if r is an integer.

(b) Conversely, suppose that $\{|X_n|^r : n \geq 1\}$ is uniformly integrable where $r \geq 1$, and show that $X_n \xrightarrow{r} X$ if $X_n \xrightarrow{\mathrm{P}} X$.

3. Let $g : [0, \infty) \to [0, \infty)$ be an increasing function satisfying $g(x)/x \to \infty$ as $x \to \infty$. Show that the sequence $\{X_n : n \geq 1\}$ is uniformly integrable if $\sup_n \mathbb{E}\{g(|X_n|)\} < \infty$.

4. Let $\{Z_n : n \geq 0\}$ be the generation sizes of a branching process with $Z_0 = 1$, $\mathbb{E}(Z_1) = 1$, $\mathrm{var}(Z_1) \neq 0$. Show that $\{Z_n : n \geq 0\}$ is not uniformly integrable.

5. Pratt's lemma. Suppose that $X_n \leq Y_n \leq Z_n$ where $X_n \xrightarrow{\mathrm{P}} X$, $Y_n \xrightarrow{\mathrm{P}} Y$, and $Z_n \xrightarrow{\mathrm{P}} Z$. If $\mathbb{E}(X_n) \to \mathbb{E}(X)$ and $\mathbb{E}(Z_n) \to \mathbb{E}(Z)$, show that $\mathbb{E}(Y_n) \to \mathbb{E}(Y)$.

6. Let $\{X_n : n \geq 1\}$ be a sequence of variables satisfying $\mathbb{E}(\sup_n |X_n|) < \infty$. Show that $\{X_n\}$ is uniformly integrable.

7.11 Problems

1. Let X_n have density function

$$f_n(x) = \frac{n}{\pi(1 + n^2x^2)}, \qquad n \geq 1.$$

With respect to which modes of convergence does X_n converge as $n \to \infty$?

2. (i) Suppose that $X_n \xrightarrow{\text{a.s.}} X$ and $Y_n \xrightarrow{\text{a.s.}} Y$, and show that $X_n + Y_n \xrightarrow{\text{a.s.}} X + Y$. Show that the corresponding result holds for convergence in rth mean and in probability, but not in distribution.

(ii) Show that if $X_n \xrightarrow{\text{a.s.}} X$ and $Y_n \xrightarrow{\text{a.s.}} Y$ then $X_nY_n \xrightarrow{\text{a.s.}} XY$. Does the corresponding result hold for the other modes of convergence?

3. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g(X_n) \xrightarrow{\mathrm{P}} g(X)$ if $X_n \xrightarrow{\mathrm{P}} X$.

4. Let $Y_1, Y_2, \ldots$ be independent identically distributed variables, each of which can take any value in $\{0, 1, \ldots, 9\}$ with equal probability $\frac{1}{10}$. Let $X_n = \sum_{i=1}^{n} Y_i\, 10^{-i}$. Show by the use of characteristic functions that X_n converges in distribution to the uniform distribution on $[0, 1]$. Deduce that $X_n \xrightarrow{\text{a.s.}} Y$ for some Y which is uniformly distributed on $[0, 1]$.

5. Let $N(t)$ be a Poisson process with constant intensity on $\mathbb{R}$.

7. (a) $r(x) = \alpha\beta x^{\beta-1}$.
(b) $r(x) = \lambda$.
(c) $r(x) = \dfrac{\lambda\alpha e^{-\lambda x} + \mu(1-\alpha)e^{-\mu x}}{\alpha e^{-\lambda x} + (1-\alpha)e^{-\mu x}}$, which approaches $\min\{\lambda, \mu\}$ as $x \to \infty$.

8. Clearly $\phi' = -x\phi$. Using this identity and integrating by parts repeatedly,

$$1 - \Phi(x) = \int_x^\infty \phi(u)\,du = -\int_x^\infty \frac{\phi'(u)}{u}\,du = \frac{\phi(x)}{x} + \int_x^\infty \frac{\phi'(u)}{u^3}\,du$$
$$= \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} - \int_x^\infty \frac{3\phi'(u)}{u^5}\,du = \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} + \frac{3\phi(x)}{x^5} - \int_x^\infty \frac{15\phi(u)}{u^6}\,du.$$

4.5 Solutions. Dependence

1. (i) As the product of non-negative continuous functions, f is non- negative and continuous. Also

$$g(x) = \tfrac{1}{2}e^{-|x|}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi x^{-2}}} e^{-\frac{1}{2}x^2y^2}\,dy = \tfrac{1}{2}e^{-|x|}$$

if $x \neq 0$, since the integrand is the $N(0, x^{-2})$ density function. It is easily seen that $g(0) = 0$, so that g is discontinuous, while

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} \tfrac{1}{2}e^{-|x|}\,dx = 1.$$

(ii) Clearly $f_Q \geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_Q(x,y)\,dx\,dy = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \cdot 1 = 1.$$

Also f_Q is the uniform limit of continuous functions on any subset of $\mathbb{R}^2$ of the form $[-M, M] \times \mathbb{R}$; hence f_Q is continuous. Hence f_Q is a continuous density function. On the other hand

$$\int_{-\infty}^{\infty} f_Q(x,y)\,dy = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n g(x - q_n),$$

where g is discontinuous at 0.
(iii) Take Q to be the set of the rationals, in some order.

2. We may assume that the centre of the rod is uniformly positioned in a square of size $a \times b$, while the acute angle between the rod and a line of the first grid is uniform on $[0, \frac{1}{2}\pi]$. If the latter angle is θ then, with the aid of a diagram, one finds that there is no intersection if and only if the centre of the rod lies within a certain inner rectangle of size $(a - r\cos\theta) \times (b - r\sin\theta)$. Hence the probability of an intersection is

$$\frac{2}{\pi ab}\int_0^{\pi/2} \{ab - (a - r\cos\theta)(b - r\sin\theta)\}\,d\theta = \frac{2r}{\pi ab}(a + b - \tfrac{1}{2}r).$$

3. (i) Let I be the indicator of the event that the first needle intersects a line, and let J be the indicator that the second needle intersects a line. By the result of Exercise (4.5.2), $\mathbb{E}(I) = \mathbb{E}(J) = 2/\pi$; hence $Z = I + J$ satisfies $\mathbb{E}(\frac{1}{2}Z) = 2/\pi$.

where $C = \{\text{two or more flies arrive in } (t, t+h]\}$ and $D = \{\text{two or more wasps arrive in } (t, t+h]\}$. This probability is no greater than $(\lambda h)(\mu h) + o(h) = o(h)$.

2. Let I be the incoming Poisson process, and let G be the process of arrivals of green insects. Matters of independence are dealt with as above. Finally,

$$\mathbb{P}\big(G(t+h) = n+1 \,\big|\, G(t) = n\big) = p\mathbb{P}\big(I(t+h) = n+1 \,\big|\, I(t) = n\big) + o(h) = p\lambda h + o(h),$$
$$\mathbb{P}\big(G(t+h) > n+1 \,\big|\, G(t) = n\big) \le \mathbb{P}\big(I(t+h) > n+1 \,\big|\, I(t) = n\big) = o(h).$$

3. Conditioning on T_1 and using the time-homogeneity of the process,

$$\mathbb{P}\big(E(t) > x \mid T_1 = u\big) = \begin{cases} \mathbb{P}\big(E(t-u) > x\big) & \text{if } u \le t, \\ 0 & \text{if } t < u \le t+x, \\ 1 & \text{if } u > t+x, \end{cases}$$

(draw a diagram to help you see this). Therefore

$$\begin{aligned}\mathbb{P}\big(E(t) > x\big) &= \int_0^\infty \mathbb{P}\big(E(t) > x \mid T_1 = u\big)\lambda e^{-\lambda u}\,du \\ &= \int_0^t \mathbb{P}\big(E(t-u) > x\big)\lambda e^{-\lambda u}\,du + \int_{t+x}^\infty \lambda e^{-\lambda u}\,du.\end{aligned}$$

You may solve the integral equation using Laplace transforms. Alternately you may guess the answer and then check that it works. The answer is $\mathbb{P}(E(t) \le x) = 1 - e^{-\lambda x}$, the exponential distribution. Actually this answer is obvious since $E(t) > x$ if and only if there is no arrival in $[t, t+x]$, an event having probability $e^{-\lambda x}$.

4. The forward equation is

$$p'_{ij}(t) = \lambda(j-1)p_{i,j-1}(t) - \lambda j p_{ij}(t), \qquad i \le j,$$

with boundary conditions $p_{ij}(0) = \delta_{ij}$, the Kronecker delta. We write $G_i(s,t) = \sum_j s^j p_{ij}(t)$, the probability generating function of $B(t)$ conditional on $B(0) = i$. Multiply through the differential equation by s^j and sum over j:

$$\frac{\partial G_i}{\partial t} = \lambda s^2 \frac{\partial G_i}{\partial s} - \lambda s \frac{\partial G_i}{\partial s},$$

a partial differential equation with boundary condition $G_i(s,0) = s^i$. This may be solved in the usual way to obtain $G_i(s,t) = g(e^{\lambda t}(1 - s^{-1}))$ for some function g. Using the boundary condition, we find that $g(1 - s^{-1}) = s^i$ and so $g(u) = (1-u)^{-i}$, yielding

$$G_i(s,t) = \frac{1}{\{1 - e^{\lambda t}(1 - s^{-1})\}^i} = \frac{(se^{-\lambda t})^i}{\{1 - s(1 - e^{-\lambda t})\}^i}.$$

The coefficient of s^j is, by the binomial series,

$$(*) \qquad p_{ij}(t) = e^{-i\lambda t}\binom{j-1}{i-1}(1 - e^{-\lambda t})^{j-i}, \qquad j \ge i,$$

as required.

Now $M_B(s)$ is non-decreasing in s, and therefore it is the value with the minus sign. The density function of B may be found by inverting the moment generating function; see Feller (1971, p. 482), who has also an alternative derivation of M_B.

As for the mean and variance, either differentiate M_B, or differentiate $(*)$. Following the latter route, we obtain the following relations involving M $(= M_B)$:

$$2\lambda M M' + M + (s - \lambda - \mu)M' = 0,$$
$$2\lambda M M'' + 2\lambda(M')^2 + 2M' + (s - \lambda - \mu)M'' = 0.$$

Set $s = 0$ to obtain $M'(0) = (\mu - \lambda)^{-1}$ and $M''(0) = 2\mu(\mu - \lambda)^{-3}$, whence the claims are immediate.

6. (i) This question is closely related to Exercise (11.3.1). With the same notation as in that solution, we have that

$$(*) \qquad Q_{n+1} = A_n + Q_n - h(Q_n)$$

where $h(x) = \min\{1, x\}$. Taking expectations, we obtain $\mathbb{P}(Q_n > 0) = \mathbb{E}(A_n)$ where

$$\mathbb{E}(A_n) = \int_0^\infty \mathbb{E}(A_n \mid S = s)\, dF_S(s) = \lambda \mathbb{E}(S) = \rho,$$

and S is a typical service time. Square $(*)$ and take expectations to obtain

$$\mathbb{E}(Q_n) = \frac{\rho(1 - 2\rho) + \mathbb{E}(A_{n+1}^2)}{2(1 - \rho)},$$

where $\mathbb{E}(A_n^2)$ is found (as above) to equal $\rho + \lambda^2 \mathbb{E}(S^2)$.
(ii) If a customer waits for time W and is served for time S, he leaves behind him a queue-length which is Poisson with parameter $\lambda(W + S)$. In equilibrium, its mean satisfies $\lambda\mathbb{E}(W + S) = \mathbb{E}(Q_n)$, whence $\mathbb{E}(W)$ is given as claimed.
(iii) $\mathbb{E}(W)$ is a minimum when $\mathbb{E}(S^2)$ is minimized, which occurs when S is concentrated at its mean. Deterministic service times minimize mean waiting time.

7. Condition on arrivals in $(t, t+h)$. If there are no arrivals, then $W_{t+h} \le x$ if and only if $W_t \le x+h$. If there is an arrival, and his service time is S, then $W_{t+h} \le x$ if and only if $W_t \le x+h-S$. Therefore

$$F(x; t + h) = (1 - \lambda h)F(x + h; t) + \lambda h \int_0^{x+h} F(x + h - s; t)\, dF_S(s) + o(h).$$

Subtract $F(x; t)$, divide by h, and take the limit as $h \downarrow 0$, to obtain the differential equation.

We take Laplace–Stieltjes transforms. Integrating by parts, for $\theta \le 0$,

$$\int_{(0,\infty)} e^{\theta x}\, dh(x) = -h(0) - \theta\{M_U(\theta) - H(0)\},$$
$$\int_{(0,\infty)} e^{\theta x}\, dH(x) = M_U(\theta) - H(0),$$
$$\int_{(0,\infty)} e^{\theta x}\, d\mathbb{P}(U + S \le x) = M_U(\theta) M_S(\theta),$$

and therefore

$$0 = -h(0) - \theta\{M_U(\theta) - H(0)\} + \lambda H(0) + \lambda M_U(\theta)\{M_S(\theta) - 1\}.$$