Student Solutions Manual

for use with

Complex Variables and Applications

Seventh Edition

Selected Solutions to Exercises in Chapters 1-7

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(*b*) In the same way, we can show that

$$\overline{z_1 z_2 \cdots z_n} = \overline{z}_1\, \overline{z}_2 \cdots \overline{z}_n \qquad (n = 2,3,\ldots).$$

This is true when $n = 2$ (Sec. 5). Assuming that it is true when $n = m$, we write

$$\begin{aligned}\overline{z_1 z_2 \cdots z_m z_{m+1}} &= \overline{(z_1 z_2 \cdots z_m) z_{m+1}} = \overline{(z_1 z_2 \cdots z_m)}\; \overline{z}_{m+1} \\ &= (\overline{z}_1 \overline{z}_2 \cdots \overline{z}_m)\overline{z}_{m+1} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_m \overline{z}_{m+1}.\end{aligned}$$

14. The identities (Sec. 5) $z\overline{z} = |z|^2$ and $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ enable us to write $|z - z_0| = R$ as

$$(z - z_0)(\overline{z} - \overline{z}_0) = R^2,$$

$$z\overline{z} - (z\overline{z}_0 + \overline{z\overline{z}_0}) + z_0\overline{z}_0 = R^2,$$

$$|z|^2 - 2\operatorname{Re}(z\overline{z}_0) + |z_0|^2 = R^2.$$

15. Since $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$, the hyperbola $x^2 - y^2 = 1$ can be written in the following ways:

$$\left(\frac{z + \overline{z}}{2}\right)^2 - \left(\frac{z - \overline{z}}{2i}\right)^2 = 1,$$

$$\frac{z^2 + 2z\overline{z} + \overline{z}^2}{4} + \frac{z^2 - 2z\overline{z} + \overline{z}^2}{4} = 1,$$

$$\frac{2z^2 + 2\overline{z}^2}{4} = 1,$$

$$z^2 + \overline{z}^2 = 2.$$

SECTION 7

1. (*a*) Since

$$\arg\left(\frac{i}{-2 - 2i}\right) = \arg i - \arg(-2 - 2i),$$

one value of $\arg\left(\dfrac{i}{-2 - 2i}\right)$ is $\dfrac{\pi}{2} - \left(-\dfrac{3\pi}{4}\right)$, or $\dfrac{5\pi}{4}$. Consequently, the principal value is

$$\frac{5\pi}{4} - 2\pi, \text{ or } -\frac{3\pi}{4}.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi\ (n = 0, \pm1, \pm2, \ldots)$. Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \qquad (n = 0 \pm 1, \pm 2, \ldots).$$

So there is no neighborhood throughout which f is analytic, and this means that $\cos \bar{z}$ is nowhere analytic.

16. *(a)* Use expression (12), Sec. 33, to write

$$\overline{\cos(iz)} = \overline{\cos(-y + ix)} = \cos y \cosh x - i \sin y \sinh x$$

and

$$\cos(i\bar{z}) = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x.$$

This shows that $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z.

(b) Use expression (11), Sec. 33, to write

$$\overline{\sin(iz)} = \overline{\sin(-y + ix)} = -\sin y \cosh x - i \cos y \sinh x$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Evidently, then, the equation $\overline{\sin(iz)} = \sin(i\bar{z})$ is equivalent to the pair of equations

$$\sin y \cosh x = 0, \quad \cos y \sinh x = 0.$$

Since $\cosh x$ is never zero, the first of these equations tells us that $\sin y = 0$. Consequently, $y = n\pi \quad (n = 0, \pm1, \pm2, \ldots)$. Since $\cos n\pi = (-1)^n \neq 0$, the second equation tells us that $\sinh x = 0$, or that $x = 0$. So we may conclude that $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = 0 + in\pi = n\pi i \quad (n = 0, \pm1, \pm2, \ldots)$.

17. Rewriting the equation $\sin z = \cosh 4$ as $\sin x \cosh y + i \cos x \sinh y = \cosh 4$, we see that we need to solve the pair of equations

$$\sin x \cosh y = \cosh 4, \quad \cos x \sinh y = 0$$

(c) Note that

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1}.$$

Referring to the final result in part *(b)*, then, we have

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-1)^{n+1}}ds = \frac{1}{2^n}\frac{1}{2\pi i}\int_C \frac{(s+1)^n}{s-1}ds = \frac{1}{2^n}2^n = 1 \qquad (n = 0,1,2,\ldots).$$

Also, since

$$\frac{(s^2-1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1},$$

we have

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s+1)^{n+1}}ds = \frac{1}{2^n}\frac{1}{2\pi i}\int_C \frac{(s-1)^n}{s+1}ds = \frac{1}{2^n}(-2)^n = (-1)^n \quad (n = 0,1,2,\ldots).$$

9. We are asked to show that

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)ds}{(s-z)^3}.$$

(a) In view of the expression for $f'(z)$ in the lemma,

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C\left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right]\frac{f(s)ds}{\Delta z}\\ &= \frac{1}{2\pi i}\int_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}f(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C\frac{f(s)ds}{(s-z)^3} &= \frac{1}{2\pi i}\int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]f(s)ds\\ &= \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}f(s)ds.\end{aligned}$$

(b) When $f(z) = \dfrac{1}{1+z^2}$, we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - \cdots \qquad (0 < |z| < 1).$$

Thus

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(0) = 0.$$

(c) If $f(z) = \dfrac{1}{z}$, it follows that $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z}$. Evidently, then,

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(1) = 2\pi i.$$

4. Let C denote the circle $|z| = 1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ $(|z| < \infty)$ enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z}\,dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!}\,dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

Now the $\dfrac{1}{z}$ in this series occurs when $n - k = -1$, or $k = n + 1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \qquad (n = 0, 1, 2, \ldots).$$

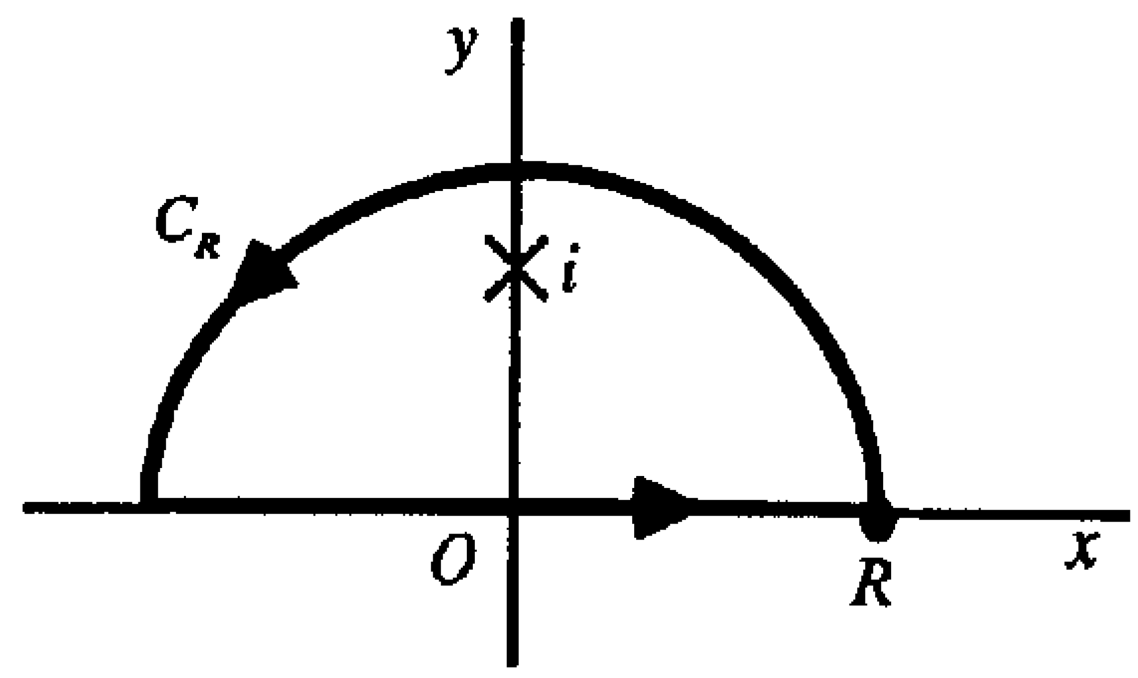
The final result in part *(a)* thus reduces to

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

Chapter 7

SECTION 72

1. To evaluate the integral $\int_0^\infty \frac{dx}{x^2+1}$, we integrate the function $f(z)=\frac{1}{z^2+1}$ around the simple closed contour shown below, where $R>1$.



We see that

$$\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1} = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=i} \frac{1}{z^2+1} = \operatorname*{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \left.\frac{1}{z+i}\right]_{z=i} = \frac{1}{2i}.$$

Thus

$$\int_{-R}^{R} \frac{dx}{x^2+1} = \pi - \int_{C_R} \frac{dz}{z^2+1}.$$

Now if z is a point on C_R,

$$|z^2+1| \geq ||z|^2 - 1| = R^2 - 1;$$

and so

$$\left| \int_{C_R} \frac{dz}{z^2+1} \right| \leq \frac{\pi R}{R^2-1} = \frac{\dfrac{\pi}{R}}{1-\dfrac{1}{R^2}} \to 0 \quad \text{as} \quad R \to \infty.$$

Finally, then

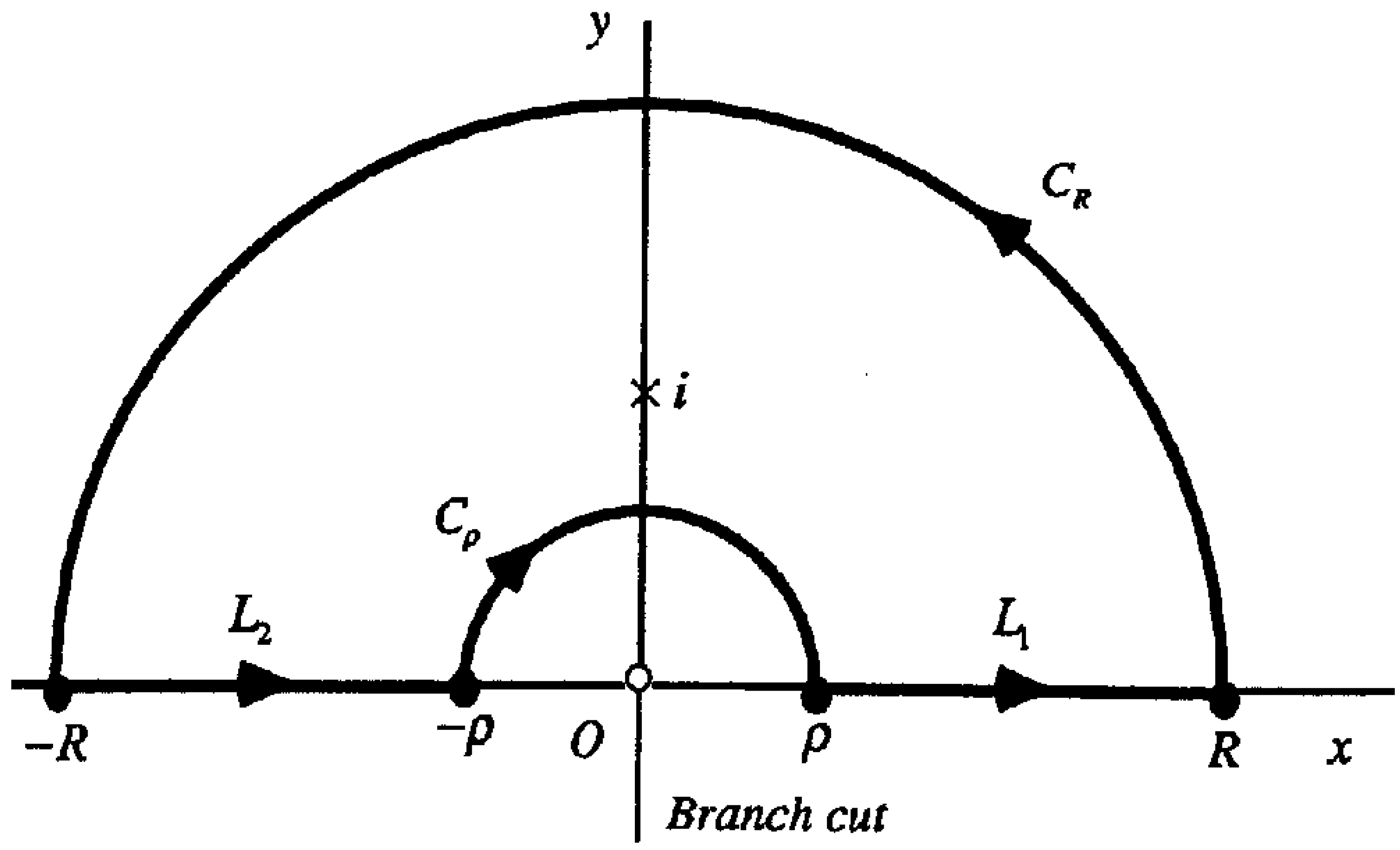
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi, \quad \text{or} \quad \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

6. *(a)* Let us first use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_\rho} f(z)\,dz=2\pi i \operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz=2\pi i \operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

Since

$$L_1: z=re^{i0}=r\ (\rho\le r\le R) \quad \text{and} \quad -L_2: z=re^{i\pi}=-r\ (\rho\le r\le R),$$

we may write

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}-i\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}=(1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}=2\pi i \operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$