Teacher's Solutions Manual

to accompany

ROGAWSKI'S CALCULUS for AP*

Early Transcendentals

Second Edition

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by

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1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a+b| = |-3+1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4.$$

Thus, $|a + b| < |a| + |b|$.

4. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

5. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

6. What is the radius of the circle with equation $(x - 9)^2 + (y - 9)^2 = 9$?

SOLUTION The circle with equation $(x - 9)^2 + (y - 9)^2 = 9$ has radius 3.

7. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

8. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

Exercises

1. Use a calculator to find a rational number r such that $|r - \pi^2| < 10^{-4}$.

SOLUTION r must satisfy $\pi^2 - 10^{-4} < r < \pi^2 + 10^{-4}$, or $9.869504 < r < 9.869705$. $r = 9.8696 = \frac{12337}{1250}$ would be one such number.

2. Which of (a)–(f) are true for $a = -3$ and $b = 2$?

(a) $a < b$ **(b)** $|a| < |b|$ **(c)** $ab > 0$

(d) $3a < 3b$ **(e)** $-4a < -4b$ **(f)** $\dfrac{1}{a} < \dfrac{1}{b}$

SOLUTION

(a) True.

(b) False, $|a| = 3 > 2 = |b|$.

(c) False, $(-3)(2) = -6 < 0$.

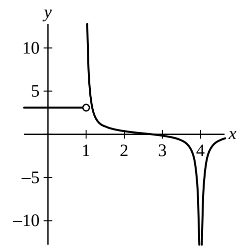
(d) True.

(e) False, $(-4)(-3) = 12 > -8 = (-4)(2)$.

(f) True.

52. $\lim_{x\to 1+} f(x) = \infty, \quad \lim_{x\to 1-} f(x) = 3, \quad \lim_{x\to 4} f(x) = -\infty$

SOLUTION



53. Determine the one-sided limits of the function $f(x)$ in Figure 6, at the points $c = 1, 3, 5, 6$.

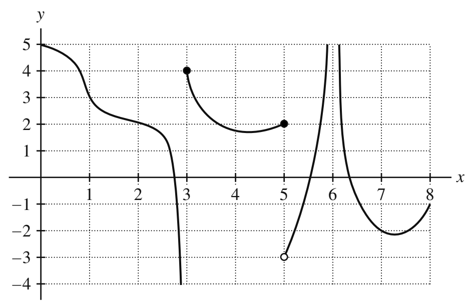


FIGURE 6 Graph of $f(x)$

SOLUTION

- $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 3$
- $\lim_{x\to 3-} f(x) = -\infty$
- $\lim_{x\to 3+} f(x) = 4$
- $\lim_{x\to 5-} f(x) = 2$
- $\lim_{x\to 5+} f(x) = -3$
- $\lim_{x\to 6-} f(x) = \lim_{x\to 6+} f(x) = \infty$

54. Does either of the two oscillating functions in Figure 7 appear to approach a limit as $x \to 0$?

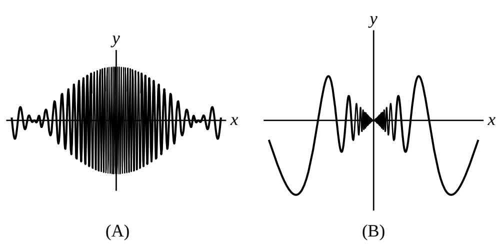


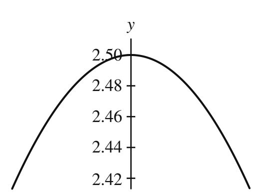
FIGURE 7

SOLUTION (A) does not appear to approach a limit as $x \to 0$; the values of the function oscillate wildly as $x \to 0$. The values of the function graphed in (B) seem to settle to 0 as $x \to 0$, so the limit seems to exist.

$\boxed{\text{GU}}$ *In Exercises 55–60, plot the function and use the graph to estimate the value of the limit.*

55. $\lim_{\theta\to 0} \dfrac{\sin 5\theta}{\sin 2\theta}$

SOLUTION



56. $\displaystyle\lim_{h\to 0} \frac{5^h - 1}{h}$

SOLUTION The difference quotient $\dfrac{5^h - 1}{h}$ has the form $\dfrac{f(a+h) - f(a)}{h}$ where $f(x) = 5^x$ and $a = 0$.

57. Apply the method of Example 6 to $f(x) = \sin x$ to determine $f'\left(\frac{\pi}{4}\right)$ accurately to four decimal places.

SOLUTION We know that

$$f'(\pi/4) = \lim_{h\to 0} \frac{f(\pi/4 + h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin(\pi/4 + h) - \sqrt{2}/2}{h}.$$

Creating a table of values of h close to zero:

h	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001
$\dfrac{\sin(\frac{\pi}{4} + h) - (\sqrt{2}/2)}{h}$	0.7074602	0.7071421	0.7071103	0.7071033	0.7070714	0.7067531

Accurate up to four decimal places, $f'(\frac{\pi}{4}) \approx 0.7071$.

58. Apply the method of Example 6 to $f(x) = \cos x$ to determine $f'\left(\frac{\pi}{5}\right)$ accurately to four decimal places. Use a graph of $f(x)$ to explain how the method works in this case.

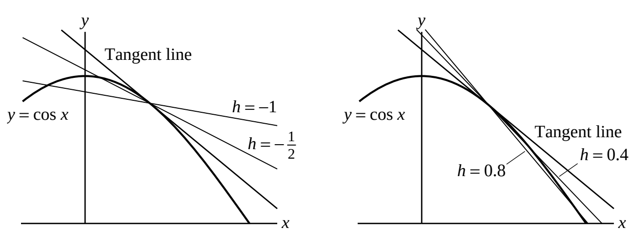
SOLUTION We know that

$$f'\left(\frac{\pi}{5}\right) = \lim_{h\to 0} \frac{f(\pi/5 + h) - f(\pi/5)}{h} = \lim_{h\to 0} \frac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}.$$

We make a chart using values of h close to zero:

h	-0.001	-0.0001	-0.00001
$\dfrac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}$	-0.587381	-0.587745	-0.587781
h	0.001	0.0001	0.00001
$\dfrac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}$	-0.588190	-0.587826	-0.587789

$f'(\frac{\pi}{5}) \approx -0.5878$. The figures shown below illustrate why this procedure works. From the figure on the left, we see that for $h < 0$, the slope of the secant line is greater (less negative) than the slope of the tangent line. On the other hand, from the figure on the right, we see that for $h > 0$, the slope of the secant line is less (more negative) than the slope of the tangent line. Thus, the slope of the tangent line must fall between the slope of a secant line with $h > 0$ and the slope of a secant line with $h < 0$.



59. For each graph in Figure 7, determine whether $f'(1)$ is larger or smaller than the slope of the secant line between $x = 1$ and $x = 1 + h$ for $h > 0$. Explain.

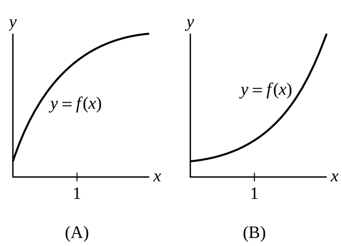


FIGURE 7

41. $y = \dfrac{x(x^2+1)}{\sqrt{x+1}}$

SOLUTION Let $y = \frac{x(x^2+1)}{\sqrt{x+1}}$. Then $\ln y = \ln x + \ln(x^2+1) - \frac{1}{2}\ln(x+1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)},$$

so

$$y' = \frac{x(x^2+1)}{\sqrt{x+1}}\left(\frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)}\right).$$

42. $y = (2x+1)(4x^2)\sqrt{x-9}$

SOLUTION Let $y = (2x+1)(4x^2)\sqrt{x-9}$. Then

$$\ln y = \ln(2x+1) + \ln 4x^2 + \ln(x-9)^{1/2} = \ln(2x+1) + \ln 4 + 2\ln x + \frac{1}{2}\ln(x-9).$$

By logarithmic differentiation

$$\frac{y'}{y} = \frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)},$$

so

$$y' = (2x+1)(4x^2)\sqrt{x-9}\left(\frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)}\right).$$

43. $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x+2)}}$

SOLUTION Let $y = \sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}}$. Then $\ln y = \frac{1}{2}[\ln(x) + \ln(x+2) - \ln(2x+1) - \ln(3x+2)]$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right),$$

so

$$y' = \frac{1}{2}\sqrt{\frac{x(x+2)}{(2x+1)(3x+2)}}\cdot\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{3}{3x+2}\right).$$

44. $y = (x^3+1)(x^4+2)(x^5+3)^2$

SOLUTION Let $y = (x^3+1)(x^4+2)(x^5+3)^2$. Then $\ln y = \ln(x^3+1) + \ln(x^4+2) + 2\ln(x^5+3)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{3x^2}{x^3+1} + \frac{4x^3}{x^4+2} + \frac{10x^4}{x^5+3},$$

so

$$y' = (x^3+1)(x^4+2)(x^5+3)^2\left(\frac{3x^2}{x^3+1} + \frac{4x^3}{x^4+2} + \frac{10x^4}{x^5+3}\right).$$

In Exercises 45–50, find the derivative using either method of Example 6.

45. $f(x) = x^{3x}$

SOLUTION Method 1: $x^{3x} = e^{3x\ln x}$, so

$$\frac{d}{dx}x^{3x} = e^{3x\ln x}(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

Method 2: Let $y = x^{3x}$. Then, $\ln y = 3x\ln x$. By logarithmic differentiation

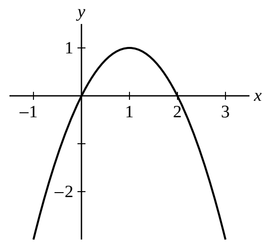
$$\frac{y'}{y} = 3x\cdot\frac{1}{x} + 3\ln x,$$

so

$$y' = y(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

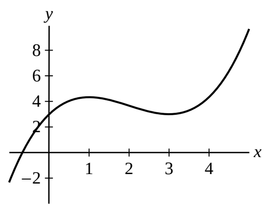
16. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$

SOLUTION Here is the graph of a function f for which $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$.



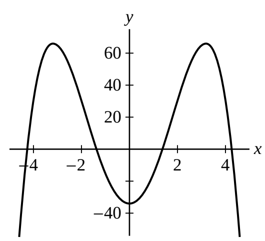
17. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.

SOLUTION Here is the graph of a function f for which $f'(x)$ is negative on $(1, 3)$ and positive elsewhere.



18. $f'(x)$ makes the sign transitions $+$, $-$, $+$, $-$.

SOLUTION Here is the graph of a function f for which f' makes the sign transitions $+, -, +, -$.



In Exercises 19–22, find all critical points of f and use the First Derivative Test to determine whether they are local minima or maxima.

19. $f(x) = 4 + 6x - x^2$

SOLUTION Let $f(x) = 4 + 6x - x^2$. Then $f'(x) = 6 - 2x = 0$ implies that $x = 3$ is the only critical point of f. As x increases through 3, $f'(x)$ makes the sign transition $+, -$. Therefore, $f(3) = 13$ is a local maximum.

20. $f(x) = x^3 - 12x - 4$

SOLUTION Let $f(x) = x^3 - 12x - 4$. Then, $f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2) = 0$ implies that $x = \pm 2$ are critical points of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-2)$ is a local maximum. On the other hand, as x increases through 2, $f'(x)$ makes the sign transition $-, +$; therefore, $f(2)$ is a local minimum.

21. $f(x) = \dfrac{x^2}{x + 1}$

SOLUTION Let $f(x) = \dfrac{x^2}{x + 1}$. Then

$$f'(x) = \frac{x(x + 2)}{(x + 1)^2} = 0$$

implies that $x = 0$ and $x = -2$ are critical points. Note that $x = -1$ is not a critical point because it is not in the domain of f. As x increases through -2, $f'(x)$ makes the sign transition $+, -$ so $f(-2) = -4$ is a local maximum. As x increases through 0, $f'(x)$ makes the sign transition $-, +$ so $f(0) = 0$ is a local minimum.

22. $f(x) = x^3 + x^{-3}$

SOLUTION Let $f(x) = x^3 + x^{-3}$. Then

$$f'(x) = 3x^2 - 3x^{-4} = \frac{3}{x^4}(x^6 - 1) = \frac{3}{x^4}(x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1) = 0$$

implies that $x = \pm 1$ are critical points of f. Though $f'(x)$ does not exist at $x = 0$, $x = 0$ is not a critical point of f because it is not in the domain of f. As x increases through -1, $f'(x)$ makes the sign transition $+, -$; therefore, $f(-1)$ is a local maximum. On the other hand, as x increases through 1, $f'(x)$ makes the sign transition $-, +$; therefore, $f(1)$ is a local minimum.

Exercises

In this exercise set, all approximations should be carried out using Newton's Method.

In Exercises 1–6, apply Newton's Method to $f(x)$ and initial guess x_0 to calculate x_1, x_2, x_3.

1. $f(x) = x^2 - 6, \quad x_0 = 2$

SOLUTION Let $f(x) = x^2 - 6$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 6}{2x_n}.$$

With $x_0 = 2$, we compute

n	1	2	3
x_n	2.5	2.45	2.44948980

2. $f(x) = x^2 - 3x + 1, \quad x_0 = 3$

SOLUTION Let $f(x) = x^2 - 3x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3}.$$

With $x_0 = 3$, we compute

n	1	2	3
x_n	2.66666667	2.61904762	2.61803445

3. $f(x) = x^3 - 10, \quad x_0 = 2$

SOLUTION Let $f(x) = x^3 - 10$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 10}{3x_n^2}.$$

With $x_0 = 2$ we compute

n	1	2	3
x_n	2.16666667	2.15450362	2.15443469

4. $f(x) = x^3 + x + 1, \quad x_0 = -1$

SOLUTION Let $f(x) = x^3 + x + 1$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1}.$$

With $x_0 = -1$ we compute

n	1	2	3
x_n	−0.75	−0.68604651	−0.68233958

5. $f(x) = \cos x - 4x, \quad x_0 = 1$

SOLUTION Let $f(x) = \cos x - 4x$ and define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - 4x_n}{-\sin x_n - 4}.$$

With $x_0 = 1$ we compute

n	1	2	3
x_n	0.28540361	0.24288009	0.24267469

3. Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$ (Figure 2).

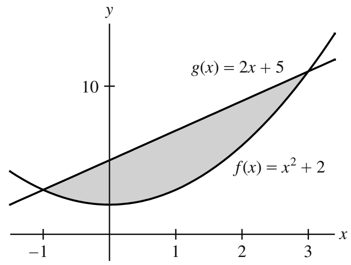


FIGURE 2

SOLUTION From the figure, we see that the graph of $g(x) = 2x + 5$ lies above the graph of $f(x) = x^2 + 2$ over the interval $[-1, 3]$. Thus, the area between the graphs is

$$\int_{-1}^{3} \left[(2x+5) - \left(x^2+2\right)\right] dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx$$
$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3}$$
$$= 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}.$$

4. Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$ (Figure 3).

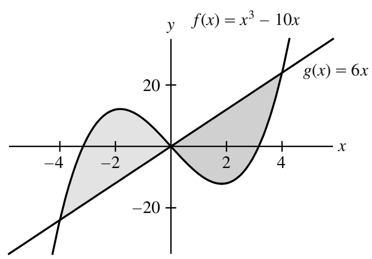


FIGURE 3

SOLUTION From the figure, we see that the graph of $f(x) = x^3 - 10x$ lies above the graph of $g(x) = 6x$ over the interval $[-4, 0]$, while the graph of $g(x) = 6x$ lies above the graph of $f(x) = x^3 - 10x$ over the interval $[0, 4]$. Thus, the area enclosed by the two graphs is

$$A = \int_{-4}^{0} \left(x^3 - 10x - 6x\right) dx + \int_{0}^{4} \left(6x - (x^3 - 10x)\right) dx$$
$$= \int_{-4}^{0} (x^3 - 16x)\, dx + \int_{0}^{4} (16x - x^3)\, dx$$
$$= \left(\frac{1}{4}x^4 - 8x^2\right)\Big|_{-4}^{0} + \left(8x^2 - \frac{1}{4}x^4\right)\Big|_{0}^{4}$$
$$= 64 + 64 = 128.$$

In Exercises 5 and 6, sketch the region between $y = \sin x$ and $y = \cos x$ over the interval and find its area.

5. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

SOLUTION Over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, the graph of $y = \cos x$ lies below that of $y = \sin x$ (see the sketch below). Hence, the area between the two curves is

$$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2} = (0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2} - 1.$$

which diverges by the Divergence Test. For the endpoint $x = 2 - e^{-1}$, the series becomes $\sum_{n=1}^{\infty}(-1)^n$, which also diverges by the Divergence Test. Thus, the series $\sum_{n=12}^{\infty} e^n(x-2)^n$ converges for $2 - e^{-1} < x < 2 + e^{-1}$ and diverges elsewhere.

34. $\sum_{n=2}^{\infty} \frac{(x+4)^n}{(n \ln n)^2}$

SOLUTION With $a_n = \frac{(x+4)^n}{(n \ln n)^2}$,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{((n+1)\ln(n+1))^2}\cdot\frac{(n\ln n)^2}{(x+4)^n}\right| = \lim_{n\to\infty}\left|(x+4)\cdot\left(\frac{n}{n+1}\cdot\frac{\ln n}{\ln(n+1)}\right)^2\right| = |x+4|$$

applying L'Hôpital's rule to evaluate the second term in the product. Thus $\rho < 1$ when $|x+4| < 1$, so the radius of convergence is 1, and the series converges absolutely on the interval $|x+4| < 1$, or $-5 < x < -3$. For the endpoint $x = -3$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{(n \ln n)^2}$, which converges by the Limit Comparison Test comparing with the convergent p-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. For the endpoint $x = -5$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n \ln n)^2}$, which converges by the Leibniz Test. Thus, the series $\sum_{n=2}^{\infty} \frac{(x+4)^n}{(n\ln n)^2}$ converges for $-5 \le x \le -3$ and diverges elsewhere.

In Exercises 35–40, use Eq. (2) to expand the function in a power series with center $c = 0$ and determine the interval of convergence.

35. $f(x) = \frac{1}{1-3x}$

SOLUTION Substituting $3x$ for x in Eq. (2), we obtain

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 3^n x^n.$$

This series is valid for $|3x| < 1$, or $|x| < \frac{1}{3}$.

36. $f(x) = \frac{1}{1+3x}$

SOLUTION Substituting $-3x$ for x in Eq. (2), we obtain

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty}(-3x)^n = \sum_{n=0}^{\infty}(-3)^n x^n.$$

This series is valid for $|-3x| < 1$, or $|x| < \frac{1}{3}$.

37. $f(x) = \frac{1}{3-x}$

SOLUTION First write

$$\frac{1}{3-x} = \frac{1}{3}\cdot\frac{1}{1-\frac{x}{3}}.$$

Substituting $\frac{x}{3}$ for x in Eq. (2), we obtain

$$\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^n};$$

Thus,

$$\frac{1}{3-x} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}.$$

This series is valid for $|x/3| < 1$, or $|x| < 3$.

38. $f(x) = \frac{1}{4+3x}$

SOLUTION The tangent line is parametrized by:

$$\ell(t) = \mathbf{r}\left(\frac{\pi}{4}\right) + t\mathbf{r}'\left(\frac{\pi}{4}\right) \tag{1}$$

We compute the vectors in the above parametrization:

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle \cos\frac{\pi}{2}, \sin\frac{3\pi}{4}\right\rangle = \left\langle 0, \frac{1}{\sqrt{2}}\right\rangle$$

$$\mathbf{r}'(t) = \frac{d}{dt}\langle \cos 2t, \sin 3t\rangle = \langle -2\sin 2t, 3\cos 3t\rangle$$

$$\Rightarrow \quad \mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle -2\sin\frac{\pi}{2}, 3\cos\frac{3\pi}{4}\right\rangle = \left\langle -2, \frac{-3}{\sqrt{2}}\right\rangle$$

Substituting the vectors in (1) we obtain the following parametrization:

$$\ell(t) = \left\langle 0, \frac{1}{\sqrt{2}}\right\rangle + t\left\langle -2, \frac{-3}{\sqrt{2}}\right\rangle = \left\langle -2t, \frac{1}{\sqrt{2}}(1-3t)\right\rangle$$

In Exercises 21–28, evaluate the integrals.

21. $\int_{-1}^{3} \left\langle 8t^2 - t, 6t^3 + t\right\rangle dt$

SOLUTION Vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component.

$$\int_{-1}^{3} 8t^2 - t\, dt = \frac{8}{3}t^3 - \frac{t^2}{2}\bigg|_{-1}^{3} = \left(72 - \frac{9}{2}\right) - \left(-\frac{8}{3} - \frac{1}{2}\right) = \frac{212}{3}$$

$$\int_{-1}^{3} 6t^3 + t\, dt = \frac{3}{2}t^4 + \frac{t^2}{2}\bigg|_{-1}^{3} = \left(\frac{243}{2} + \frac{9}{2}\right) - \left(\frac{3}{2} + \frac{1}{2}\right) = 124$$

Therefore,

$$\int_{-1}^{3} \left\langle 8t^2 - t, 6t^3 + t\right\rangle dt = \left\langle \int_{-1}^{3} 8t^2 - t\, dt, \int_{-1}^{3} 6t^3 + t\, dt\right\rangle = \left\langle \frac{212}{3}, 124\right\rangle$$

22. $\int_0^1 \left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2}\right\rangle ds$

SOLUTION The vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component. For the second integral we use the substitution $t = 1 + s^2$, $dt = 2s\,ds$. We get:

$$\int_0^1 \frac{ds}{1+s^2} = \tan^{-1}(s)\bigg|_0^1 = \tan^{-1}(1) - \tan^1(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^1 \frac{s}{1+s^2}\, ds = \int_1^2 \frac{1}{t}\left(\frac{dt}{2}\right) = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln t\bigg|_1^2 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Therefore,

$$\int_0^1 \left\langle \frac{1}{1+s^2}, \frac{s}{1+s^2}\right\rangle ds = \left\langle \int_0^1 \frac{ds}{1+s^2}, \int_0^1 \frac{s\,ds}{1+s^2}\right\rangle = \left\langle \frac{\pi}{4}, \frac{1}{2}\ln 2\right\rangle$$

23. $\int_{-2}^{2} (u^3\mathbf{i} + u^5\mathbf{j})\, du$

SOLUTION The vector-valued integration is defined via componentwise integration. Thus, we first compute the integral of each component.

$$\int_{-2}^{2} u^3\, du = \frac{u^4}{4}\bigg|_{-2}^{2} = \frac{16}{4} - \frac{16}{4} = 0$$

$$\int_{-2}^{2} u^5\, du = \frac{u^6}{6}\bigg|_{-2}^{2} = \frac{64}{6} - \frac{64}{6} = 0$$