PART A
Ordinary Differential Equations (ODEs)

Chap. 1 First-Order ODEs

Sec. 1.1 Basic Concepts. Modeling

To get a good start into this chapter and this section, quickly **review your basic calculus**. Take a look at the front matter of the textbook and see a review of the main differentiation and integration formulas. Also, Appendix 3, pp. A63–A66, has useful formulas for such functions as exponential function, logarithm, sine and cosine, etc. The beauty of ordinary differential equations is that the subject is quite systematic and has different methods for different types of ordinary differential equations, as you shall learn. Let us discuss some Examples of Sec. 1.1, pp. 4–7.

Example 2, p. 5. Solution by Calculus. Solution Curves. To solve the first-order ordinary differential equation (ODE)

$$y' = \cos x$$

means that we are looking for a function whose derivative is $\cos x$. Your first answer might be that the desired function is $\sin x$, because $(\sin x)' = \cos x$. But your answer would be incomplete because also $(\sin x + 2)' = \cos x$, since the derivative of 2 and of any constant is 0. Hence the complete answer is $y = \cos x + c$, where c is an arbitrary constant. As you vary the constants you get an infinite family of solutions. Some of these solutions are shown in **Fig. 3**. The lesson here is that you should never forget your constants!

Example 4, pp. 6–7. Initial Value Problem. In an initial value problem (IVP) for a first-order ODE we are given an ODE, here $y' = 3y$, and an initial value condition $y(0) = 5.7$. For such a problem, the first step is to solve the ODE. Here we obtain $y(x) = ce^{3x}$ as shown in **Example 3**, p. 5. Since we also have an initial condition, we must substitute that condition into our solution and get $y(0) = ce^{3\cdot 0} = ce^0 = c \cdot 1 = c = 5.7$. Hence the complete solution is $y(x) = 5.7e^{3x}$. The lesson here is that for an initial value problem you get a unique solution, also known as a particular solution.

Modeling means that you interpret a physical problem, set up an appropriate mathematical model, and then try to solve the mathematical formula. Finally, you have to interpret your answer. Examples 3 (exponential growth, exponential decay) and 5 (radioactivity) are examples of modeling problems. Take a close look at **Example 5**, p. 7, because it outlines all the steps of modeling.

Problem Set 1.1. Page 8

3. Calculus. From Example 3, replacing the independent variable t by x we know that $y' = 0.2y$ has a solution $y = 0.2ce^{0.2x}$. Thus by analogy, $y' = y$ has a solution

$$1 \cdot ce^{1 \cdot x} = ce^x,$$

where c is an arbitrary constant.

Another approach (to be discussed in details in Sec. 1.3) is to write the ODE as

$$\frac{dy}{dx} = y,$$

and then by algebra obtain

$$dy = y\,dx, \quad \text{so that} \quad \frac{1}{y}dy = dx.$$

Integrate both sides, and then apply exponential functions on both sides to obtain the same solution as above

$$\int \frac{1}{y}dy = \int dx, \qquad \ln|y| = x + c, \qquad e^{\ln|y|} = e^{x+c}, \qquad y = e^x \cdot e^c = c^* e^x,$$

(where $c^* = e^c$ is a constant).

The technique used is called **separation of variables** because we separated the variables, so that y appeared on one side of the equation and x on the other side before we integrated.

7. Solve by integration. Integrating $y' = \cosh 5.13x$ we obtain (chain rule!) $y = \int \cosh 5.13x\,dx = \frac{1}{5.13}(\sinh 5.13x) + c$. Check: Differentiate your answer:

$$\left(\tfrac{1}{5.13}(\sinh 5.13x) + c\right)' = \tfrac{1}{5.13}(\cosh 5.13x) \cdot 5.13 = \cosh 5.13x, \text{ which is correct.}$$

11. Initial value problem (IVP). (a) Differentiation of $y = (x + c)e^x$ by product rule and definition of y gives

$$y' = e^x + (x + c)e^x = e^x + y.$$

But this looks precisely like the given ODE $y' = e^x + y$. Hence we have shown that indeed $y = (x + c)e^x$ is a solution of the given ODE. **(b)** Substitute the initial value condition into the solution to give $y(0) = (0 + c)e^0 = c \cdot 1 = \frac{1}{2}$. Hence $c = \frac{1}{2}$ so that the answer to the IVP is

$$y = (x + \tfrac{1}{2})e^x.$$

(c) The graph intersects the x-axis at $x = 0.5$ and shoots exponentially upward.

Three cases appear, as for those other ODEs, and Fig. 48, p. 73, gives an idea of what kind of solution we can expect. In some cases $x = 0$ must be excluded (when we have a power with a negative exponent), and in other cases the solutions are restricted to positive values for the independent variable x; this happens when a logarithm or a root appears (see Example 1, p. 71). Note further that the auxiliary equation for determining exponents m in $y = x^m$ is

$$m(m-1) + a\,m + b = 0, \quad \text{thus } m^2 + (a-1)\,m + b = 0,$$

with $a - 1$ as the coefficient of the linear term. Here the ODE is written

$$x^2y'' + axy' + by = 0, \tag{1}$$

which is no longer in the standard form with y'' as the first term.

Whereas constant-coefficient ODEs are basic in mechanics and electricity, Euler–Cauchy equations are less important. A typical application is shown on p. 73.

In summary, we can say that the key approach to solving the Euler–Cauchy equation is the auxiliary equation $m(m-1) + am + b = 0$. From this most of the material develops.

Problem Set 2.5. Page 73

3. General solution. Double root (Case II). Problems 2–11 are solved, as explained in the text, by determining the roots of the auxiliary equation (3). The ODE $5x^2y'' + 23xy' + 16.2y = 0$ has the auxiliary equation

$$5m(m-1) + 23m + 16.2 = 5m^2 + 18m + 16.2 = 5[(m+1.8)(m+1.8)] = 0.$$

According to (6), p. 72, a general solution for positive x is

$$y = (c_1 + c_2 \ln x)x^{-1.8}.$$

5. Complex roots. The ODE $4x^2y'' + 5y = 0$ has the auxiliary equation

$$4m(m-1) + 5 = 4m^2 - 4m + 5 = 4\big(m - \left(\tfrac{1}{2} + i\right)\big)\big(m - \left(\tfrac{1}{2} - i\right)\big) = 0.$$

A basis of complex solutions is $x^{(1/2)+i}, x^{(1/2)-i}$. From it we obtain real solutions by a trick that introduces exponential functions, namely, by first writing (Euler's formula!)

$$x^{(1/2)+i} = x^{1/2}x^{\pm i} = x^{1/2}e^{\pm i \ln x} = x^{1/2}(\cos(\ln x) \pm i \sin(\ln x))$$

and then taking linear combinations to obtain a real basis of solutions

$$\sqrt{x}\cos(\ln x) \qquad \text{and} \qquad \sqrt{x}\sin(\ln x)$$

for positive x or writing $\ln |x|$ if we want to admit all $x \neq 0$.

7. Real roots. The ODE is in D-notation, with D the differential operator from Sec. 2.3. In regular notation we have

$$(x^2D^2 - 4xD + 6I)y = x^2D^2y - 4xDy - 6Iy = x^2y'' - 4xy' + 6y = 0.$$

Using the method of Example 1 of the text and determing the roots of the auxiliary equation (3) we obtain

$$m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3) = 0$$

and from this the general solution $y = c_1x^2 + c_2x^3$ valid for all x follows.

For $\lambda_2 = -\sqrt{2}i$, we obtain an eigenvector as follows:

$$\begin{bmatrix} \sqrt{2}i & 1 & 0 \\ -1 & \sqrt{2}i & 1 \\ 0 & -1 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the following system of linear equations:

$$\begin{aligned} \sqrt{2}ix_1 + \quad x_2 \qquad\quad &= 0 \quad \text{so that} \quad x_2 = -\sqrt{2}ix_1. \\ -x_1 + \sqrt{2}x_2 + \quad x_3 &= 0, \\ -x_2 + \sqrt{2}ix_3 &= 0. \end{aligned}$$

Substituting $x_3 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation gives us

$$\begin{aligned} -x_1 + \sqrt{2}x_2 + x_3 &= -x_1 + (\sqrt{2}i)(-\sqrt{2}i)x_1 + x_3 \\ &= -x_1 + 2x_1 + x_3 = x_1 + x_3 = 0 \quad \text{hence} \quad x_1 = -x_3. \end{aligned}$$

[Note that, to simplify the coefficient of the x_1-term, we used that $(\sqrt{2}i)(-\sqrt{2}i) = -(\sqrt{2})(\sqrt{2}) \cdot (\sqrt{-1})(\sqrt{-1}) = -(2)(-1) = -2$, where $i = \sqrt{-1}$.] Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = -\sqrt{2}i$. Thus the eigenvector for $\lambda_2 = -\sqrt{2}i$ is

$$[x_1 \quad x_2 \quad x_3]^{\mathsf{T}} = [1 \quad -\sqrt{2}i \quad -1]^{\mathsf{T}}.$$

For $\lambda_3 = \sqrt{2}i$, we obtain the following system of linear equations:

$$\begin{aligned} -\sqrt{2}ix_1 + \quad x_2 \qquad\quad &= 0 \quad \text{so that} \quad x_2 = \sqrt{2}ix_1. \\ -x_1 - \sqrt{2}ix_2 + \quad x_3 &= 0, \\ -x_2 - \sqrt{2}ix_3 &= 0 \quad \text{so that} \quad x_2 = -\sqrt{2}ix_1. \end{aligned}$$

Substituting $x_2 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation

$$x_1 = -\sqrt{2}x_2 + x_3 = -\sqrt{2}i\sqrt{2}ix_1 + x_3 = 2x_1 + x_3, \text{ hence } x_1 = -x_3.$$

(Another way to see this is to note that, $x_2 = \sqrt{2}ix_1$ and $x_2 = -\sqrt{2}ix_3$, so that $\sqrt{2}ix_1 = -\sqrt{2}ix_3$ and hence $x_1 = -x_3$.) Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = \sqrt{2}i$. Thus the eigenvector for $\lambda_3 = \sqrt{2}i$ is $[1 \quad \sqrt{2}i \quad -1]^{\mathsf{T}}$, as was to be expected from before. For more complicated calculations, you might want to use Gaussian elimination (to be discussed in Sec. 7.3).

Together, we obtain the general solution

$$\mathbf{y} = c_1^* \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{0t} + c_2^* \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it} + c_3^* \begin{bmatrix} 1 \\ \sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it}$$

Sec. 6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

Do not confuse differentiation of *transforms* (Sec. 6.6) with differentiation of *functions* $f(t)$ (Sec. 6.2). The latter is basic to the whole transform method of solving ODEs. The present discussion on differentiation of transforms adds just another method of obtaining transforms and inverses. It completes some of the theory for Sec. 6.1 as shown on p. 238.

Also, solving ODEs with variable coefficients by the present method is restricted to a few such ODEs, of which the most important one is perhaps Laguerre's ODE (p. 240). This is because its solutions, the Laguerre polynomials, are orthogonal [by Team Project 14(b) on p. 504]. Our hard work has paid off and we have built such a large repertoire of techniques for dealing with Laplace transforms that we may have several ways of solving a problem. This is illustrated in the four solution methods in **Prob. 3**. The choice depends on what we notice about how the problem is put together, and there may be a preferred method as indicated in **Prob. 15.**

Problem Set 6.6. Page 241

3. Differentiation, shifting. We are given that $f(t) = \frac{1}{2}te^{-3t}$ and asked to find $\mathcal{L}(\frac{1}{2}te^{-3t})$. For better understanding we show that there are four ways to solve this problem.

Method 1. Use first shifting (Sec. 6.1). From Table 6.1, Sec. 6.1, we know that

$$\tfrac{1}{2}t \qquad \text{has the transform} \qquad \frac{\frac{1}{2}}{s^2}.$$

Now we apply the first shifting theorem (Theorem 2, p. 208) to conclude that

$$\left(\frac{1}{2}t\right)(e^{-3t}) \qquad \text{has the transform} \qquad \frac{\frac{1}{2}}{(s-(-3))^2}.$$

Method 2. Use differentiation, the preferred method of this section (Sec. 6.6). We have

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

so that by (1), in the present section, we have

$$\mathcal{L}(tf) = \mathcal{L}\left(\frac{1}{2}te^{-3t}\right) = -\left(\frac{1}{2}\frac{1}{s+3}\right)' = -\left(-\frac{1}{2}\frac{1}{(s+3)^2}\right) = \frac{1}{2}\frac{1}{(s+3)^2} = \frac{\frac{1}{2}}{(s-(-3))^2}.$$

Method 3. Use of subsidiary equation (Sec. 6.2). As a third method, we write $g = \frac{1}{2}te^{-3t}$. Then $g(0) = 0$ and by calculus

$$g' = \tfrac{1}{2}e^{-3t} - 3\left(\tfrac{1}{2}te^{-3t}\right) = \tfrac{1}{2}e^{-3t} - 3g. \tag{A}$$

The subsidiary equation with $G = \mathcal{L}(g)$ is

$$sG = \frac{\frac{1}{2}}{s+3} - 3G, \qquad (s+3)G = \frac{\frac{1}{2}}{s+3}, \qquad G = \frac{\frac{1}{2}}{(s+3)^2}.$$

Method 4. Transform problem into second-order initial value problem (Sec. 6.2) and solve it. As a fourth method, an unnecessary detour, differentiate (A) to get a second-order ODE:

$$g'' = -\tfrac{3}{2}e^{-3t} - 3g' \qquad \text{with initial conditions} \qquad g(0) = 0,\ g'(0) = \tfrac{1}{2}$$

and solve the IVP by the Laplace transform, obtaining the same transform as before.

$\mathbf{r} = \overrightarrow{OP} = [8, 6, 0]$. From these data the formula (9), p. 372, provides the solution of the equation and formula (2**), p. 370, expands the cross product. Hence the desired velocity (vector) is

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 20 & 0 \\ 8 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 20 & 0 \\ 6 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 8 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 20 \\ 8 & 6 \end{vmatrix} \mathbf{k},$$

which we solve by Theorem 2 (b) on p. 297 in Sec. 7.7 and (1), p. 291 in Sec. 7.6. From Theorem 2 we immediately know that the first two determinants in front of **i** and **j** are 0. The last determinant gives us

$$\begin{vmatrix} 0 & 20 \\ 8 & 6 \end{vmatrix} = 0 \cdot 6 - 20 \cdot 8 = -160.$$

Thus the desired velocity is $\mathbf{v} = [0, 0, -160]$. The speed is the length of the velocity vector **v**, that is, $|\mathbf{v}| = \sqrt{(-160)^2} = 160$.

11. Vector product (Cross Product). Anticommutativity. From the given vectors $\mathbf{a} = [2, 1, 0]$ and $\mathbf{b} = [-3, 2, 0]$ we calculate the vector product or cross product by (2**), p. 370, denote it by vector **v**, and get

$$\begin{aligned} \mathbf{v} = \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} &= \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot 0 - 0 \cdot 2)\mathbf{i} - (2 \cdot 0 - 0 \cdot (-3))\mathbf{j} + (2 \cdot 2 - 1 \cdot (-3))\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 7\mathbf{k} = [0, \quad 0, \quad 7]. \end{aligned}$$

Similarly, if we denote the second desired vector product by **w**, then

$$\begin{aligned} \mathbf{w} = \mathbf{c} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} &= \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot 0 - 0 \cdot 2)\mathbf{i} - ((-3) \cdot 0 - 0 \cdot 2)\mathbf{j} + ((-3) \cdot 2 - 2 \cdot 1)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} - 7\mathbf{k} = [0, \quad 0, \quad -7]. \end{aligned}$$

Finally, the inner product or dot product in components as given by (2) on p. 361 gives us

$$[2, 1, 0] \bullet [-3, 2, 0] = 2 \cdot (-3) + 1 \cdot 2 + 0 \cdot 0 = -6 + 2 + 0 = -4.$$

Comments. We could have computed **v** by (2*) on p. 369 instead of (2**). The advantage of (2**) is that it is easier to remember. In that same computation, we could have used Theorem 2(c) on p. 297 of Sec. 7.7 to immediately conclude the first second-order determinant, having a row of zeros, has a value of zero. Similarly for the second-order determinant. For the second cross product, we could have used (6) in Theorem 1(c) on p. 370 and gotten quickly that $\mathbf{w} = \mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c} = -\mathbf{v} = -[0, \quad 0, \quad 7] = [0, \quad 0, \quad -7]$. Since $\mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c}$, the cross product is not commutative but *anticommutative*. This is much better than with matrix multiplication which, in general, was neither commutative nor anticommutative. We could have used these comments to simplify our calculations, but we just wanted to show you that the straightforward approach works.

first over x, the limits of integration must be expressed in terms of y. Thus $y = x^2$ becomes $x = \sqrt{y}$ (see (4) and Fig. 230, p. 428, for conceptual understanding):

$$\begin{aligned}
\int_0^1 \int_y^{\sqrt{y}} x^3\, dx\, dy &= \int_{y=0}^1 \left(\int_{x=y}^{\sqrt{y}} x^3\, dx \right) dy \\
&= \int_{y=0}^1 \left[\frac{x^4}{4} \right]_{x=y}^{x=\sqrt{y}} dy \\
&= \int_0^1 \left(\frac{y^2}{4} - \frac{y^4}{4} \right) dy \\
&= \frac{1}{4} \left(\int_0^1 y^2\, dy - \int_0^1 y^4\, dy \right) \\
&= \frac{1}{4} \left(\left[\frac{y^3}{3} \right]_0^1 - \left[\frac{y^5}{5} \right]_0^1 \right) \\
&= \frac{1}{4} \left(\frac{1}{3} - \frac{1}{5} \right) \\
&= \frac{1}{4} \left(\frac{5}{15} - \frac{3}{15} \right) \\
&= \frac{1}{4} \cdot \frac{2}{15} = \frac{2}{60} \\
&= \frac{1}{30} = 0.03333.
\end{aligned}$$

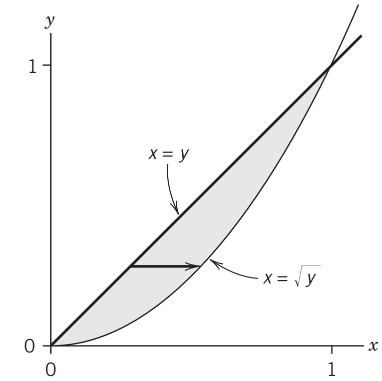


Fig. 10.3(b). Integrating first in x-direction and then in y-direction

5. Electrostatic potential in a disk. Use of formula (20), p. 591. We are given that the boundary values are

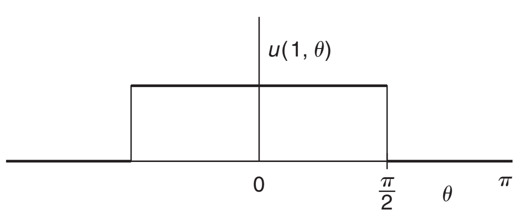
$$u(1,\theta) = f(\theta) = 220 \quad \text{if} \quad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \qquad \text{and} \qquad 0 \qquad \text{otherwise.}$$

We sketch as shown below. We note that the period p of $f(\theta)$ is 2π, which is reasonable as we are dealing with a disk $r < R = 1$. Hence the period $p = 2L = 2\pi$ so that $L = \pi$. Furthermore, $f(\theta)$ is an even function. Hence we use (6*), p. 486, to compute the coefficients of the Fourier series as required by (20), p. 591. Since $f(\theta)$ is even, the coefficients b_n in (20) are 0, that is, $f(\theta)$ is not represented by any sine terms but only cosine terms.

We compute

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi/2} f(\theta)\,d\theta \qquad\qquad \left[\text{since } f(\theta) = 0,\ \theta \geq \frac{\pi}{2}\right] \\
&= \frac{1}{\pi}\int_0^{\pi/2} 220\,d\theta \\
&= \frac{220}{\pi}\big[\theta\big]_0^{\pi/2} \\
&= \frac{220}{\pi}\cdot\frac{\pi}{2} = 110. \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^{\pi/2} f(\theta)\cos\frac{n\pi\theta}{L}\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} 220\cdot\cos\,n\theta\,d\theta = \frac{2}{\pi}\cdot 220\int_0^{\pi/2}\cos\,n\theta\,d\theta \\
&= \frac{440}{\pi}\left[\frac{\sin n\theta}{n}\right]_0^{\pi/2} = \frac{440}{n\pi}\sin\frac{n\pi}{2}.
\end{aligned}$$

For even n this is 0. For $n = 1, 5, 9, \cdots$ this equals $440/(n\pi)$, and for $n = 3, 7, 11, \cdots$ it equals $-440/(n\pi)$. Writing the Fourier series out gives the answer shown on p. A33.



Sec. 12.10 Prob. 5. Boundary potential

11. Semidisk. The idea sketched in the answer on p. A34 is conceived by symmetry. Proceeding in that way will guarantee that, on the horizontal axis, we have potential 0. This can be confirmed by noting