PART D
Complex Analysis

Chap. 13 Complex Numbers and Functions. Complex Differentiation

Complex numbers appeared in the textbook before in different topics. Solving linear homogeneous ODEs led to characteristic equations, (3), p. 54 in Sec. 2.2, with complex numbers in Example 5, p. 57, and Case III of the table on p. 58. Solving algebraic eigenvalue problems in Chap. 8 led to characteristic equations of matrices whose roots, the eigenvalues, could also be complex as shown in Example 4, p. 328. Whereas, in these type of problems, complex numbers appear almost naturally as complex roots of polynomials (the simplest being $x^2 + 1 = 0$), *it is much less immediate to consider* ***complex analysis****—the systematic study of complex numbers, complex functions, and "complex" calculus.* Indeed, complex analysis will be the direction of study in Part D. The area has important engineering applications in electrostatics, heat flow, and fluid flow. Further motivation for the study of complex analysis is given on p. 607 of the textbook.

We start with the basics in Chap. 13 by reviewing complex numbers $z = x + yi$ in Sec. 13.1 and introducing complex integration in Sec.13.3. Those functions that are differentiable in the complex, on some domain, are called **analytic** and will form the basis of complex analysis. Not all functions are analytic. This leads to the most important topic of this chapter, the **Cauchy–Riemann equations** (1), p. 625 in Sec. 13.4, which allow us to test whether a function is analytic. They are very short but you have to remember them! The rest of the chapter (Secs. 13.5–13.7) is devoted to elementary complex functions (exponential, trigonometric, hyperbolic, and logarithmic functions).

Your knowledge and understanding of real calculus will be useful. Concepts that you learned in real calculus carry over to complex calculus; however, be aware that *there are* ***distinct differences between real calculus and complex analysis*** that we clearly mark. For example, whereas the real equation $e^x = 1$ has only one solution, its complex counterpart $e^z = 1$ has infinitely many solutions.

Sec. 13.1 Complex Numbers and Their Geometric Representation

Much of the material may be familiar to you, but we start from scratch to assure everyone starts at the same level. This section begins with the four basic algebraic operations of complex numbers (addition, subtraction, multiplication, and division). Of these, the one that perhaps differs most from real numbers is **division** (or **forming a quotient**). *Thus make sure that you remember how to calculate the quotient of two complex numbers as given in equation* (7), **Example 2**, p. 610, and **Prob. 3**. In (7) we take the number z_2 from the denominator and form its complex conjugate $\bar{z}_2$ and a new quotient $\bar{z}_2/\bar{z}_2$. We multiply the given quotient by this new quotient $\bar{z}_2/\bar{z}_2$ (which is equal to 1 and thus allowed):

$$z = \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot 1 = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2},$$

which we multiply out, recalling that $i^2 = -1$ [see (5), p. 609]. The final result is a complex number in a form that allows us to separate its real ($\operatorname{Re} z$) and imaginary ($\operatorname{Im} z$) parts. Also remember that $1/i = -i$ (see **Prob. 1**), as it occurs frequently. We continue by defining the **complex plane** and use it to graph complex numbers (note Fig. 318, p. 611, and Fig. 322, p. 612). We use equation (8), p. 612, to go from complex to real.

Problem Set. 13.1. Page 612

1. Powers of *i*. We compute the various powers of i by the rules of addition, subtraction, multiplication, and division given on pp. 609–610 of the textbook. We have formally that

$$\text{(I1)} \qquad \begin{aligned} i^2 &= i\,i \\ &= (0,1)(0,1) && \text{[by (1), p. 609]} \\ &= (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0) && \text{[by (3), p. 609]} \\ &= (0-1, 0+0) && \text{(arithmetic)} \\ &= (-1, 0) \\ &= -1 && \text{[by (1)]}, \end{aligned}$$

where in (3), that is, *multiplication of complex numbers*, we used $x_1 = 0$, $x_2 = 0$, $y_1 = 1$, $y_2 = 1$.

$$\text{(I2)} \qquad i^3 = i^2 i = (-1)\cdot i = -i.$$

Here we used (I1) in the second equality. To get (I3), we apply (I2) twice:

$$\text{(I3)} \qquad i^4 = i^2 i^2 = (-1)\cdot(-1) = 1.$$

$$\text{(I4)} \qquad i^5 = i^4 i = 1 \cdot i = i,$$

and the pattern repeats itself as summarized in the table below.
We use (7), p. 610, in the following calculation:

$$\text{(I5)} \qquad \frac{1}{i} = \frac{1}{i}\frac{\bar{i}}{\bar{i}} = \frac{1}{i}\frac{(-i)}{(-i)} = \frac{(1+0i)(0-i)}{(0+i)(0-i)} = \frac{1\cdot 0 + 0 \cdot 1}{0^2+1^2} + i\,\frac{0\cdot 0 - 1\cdot 1}{0^2+1^2} = 0 - i = -i.$$

Thus

$$e^{z^2} = e^{x^2-y^2+i2xy} = e^{x^2-y^2}e^{i2xy} \qquad \text{[by (3), p. 630].}$$

Now

$$e^{i2xy} = \cos(2xy) + i\sin(2xy) \qquad \text{[by (1), p. 630; (5), p. 631].}$$

Putting it together

$$\begin{aligned} e^{z^2} &= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)] \\ &= e^{x^2-y^2}\cos 2xy + i(e^{x^2-y^2}\sin 2xy). \end{aligned}$$

Hence

$$\operatorname{Re}\left[\exp(z^2)\right] = e^{x^2-y^2}\cos 2xy; \qquad \operatorname{Im}\left[\exp(z^2)\right] = e^{x^2-y^2}\sin 2xy,$$

as given on p. A36 of the textbook.

19. Equation. To solve

(A) $$e^z = 1$$

we set $z = x + iy$. Then

$$\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) && \text{[by (5), p. 631]} \\ &= e^x\cos y + ie^x\sin y \\ &= 1 && \text{[by (A)]} \\ &= 1 + i\cdot 0. \end{aligned}$$

Equate the real and imaginary parts on both sides to obtain

$$\text{(B)} \quad \operatorname{Re}(e^z) = e^x\cos\, y = 1, \qquad \text{(C)} \quad \operatorname{Im}(e^z) = e^x\sin\, y = 0.$$

Since $e^x > 0$ but the product in (C) must equal zero requires that

$$\sin y = 0 \quad \text{which means that} \quad \text{(D)} \quad y = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Since the product in (B) is positive, $\cos y$ has to be positive. If we look at (D), we know that $\cos y$ is -1 for $y = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ but $+1$ for $y = 0, \pm 2\pi, \pm 4\pi, \ldots$. Hence (B) and (D) give

(E) $$y = 0, \pm 2\pi, \pm 4\pi, \ldots.$$

Since (B) requires that the product be equal to 1 and the cosine for the values of y in (E) is 1, we have $e^x = 1$. Hence

(F) $$x = 0.$$

Then (E) and (F) together yield

$$x = 0 \quad y = 0, \pm 2\pi, \pm 4\pi, \ldots\ ,$$

and the desired solution to (A) is

$$z = x + yi = \pm 2n\pi i, \qquad n = 0, 1, 2, \ldots.$$

Note that (A), being complex, has infinitely many solutions in contrast to the same equation in real, which has only one solution.

so that the final result is

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{2n+2}{2n+1} = \lim_{n\to\infty}\frac{\frac{2n+2}{n}}{\frac{2n+1}{n}} = \lim_{n\to\infty}\frac{2+\frac{2}{n}}{2+\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty}\left(2+\frac{2}{n}\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)} = \frac{2+0}{2+0} = \frac{2}{2} = 1 = R$$

Thus the series converges in the open disk $|z-2i\,| < 1$ of radius $R = 1$ and center $2i$.

Sec. 15.3 Functions Given by Power Series

We now give some theoretical foundations for power series and show how we can develop a new power series from an existing one. This can be done in two ways. We can **differentiate** a power series term by term without changing the radius of convergence (Theorem 3, p. 687, Example 1, p. 688, Prob. 5). Similarly, we can **integrate** (Theorem 4, p. 688, Prob. 9). Most importantly, Theorem 5, p. 688, gives the reason why power series are of central importance in complex analysis since power series are analytic and so are "differentiated" power series (with the radius of convergence preserved).

Problem Set 15.3. Page 689

5. Radius of convergence by differentiation: Theorem 3, p. 687. We start with the geometric series

$$\text{(A)} \qquad g(z) = \sum_{n=0}^{\infty}\left(\frac{z-2i}{2}\right)^n = 1 + \frac{z-2i}{2} + \left(\frac{z-2i}{2}\right)^2 + \left(\frac{z-2i}{2}\right)^3 + \cdots .$$

Using Example 1, p. 680, of Sec. 15.2, we know that it converges for

$$\frac{|\,z-2i\,|}{2} < 1 \qquad \text{and thus for} \qquad |\,z-2i\,| < 2.$$

Theorem 3, p. 687, allows us to differentiate the series in (A), termwise, with the radius of convergence preserved. Hence we get

$$\text{(B)} \qquad g'(z) = 0 + \frac{1}{2} + 2\left(\frac{z-2i}{2}\right)\cdot\frac{1}{2} + 3\left(\frac{z-2i}{2}\right)^2\cdot\frac{1}{2} + \cdots$$

$$= \sum_{n=1}^{\infty}\frac{n(z-2i)^{n-1}}{2^n} \qquad \text{where} \qquad |\,z-2i\,| < 2.$$

Note that we sum from $n = 1$ because the term for $n = 0$ is 0.

Applying Theorem 3 to (B) yields

$$\text{(C)} \qquad g''(z) = \sum_{n=2}^{\infty}\frac{n(n-1)(z-2i)^{n-2}}{2^n} \qquad \text{where} \qquad |\,z-2i\,| < 2.$$

From (C) it follows that

$$\text{(D)} \qquad (z-2i)^2 g''(z) = \sum_{n=2}^{\infty}\frac{n(n-1)(z-2i)^n}{2^n}$$

$$= \sum_{n=2}^{\infty} n(n-1)\left(\frac{z-2i}{2}\right)^n \qquad \text{where} \qquad |\,z-2i\,| < 2.$$

But (D) is precisely the given series.

This is the y-axis. Then

$$|z-c|^2 = |z+c|^2 = y^2 + c^2, \qquad \frac{|z-c|}{|z+c|} = 1, \qquad \operatorname{Ln} 1 = 0.$$

This shows that the y-axis has potential 0.

We can now continue with (B), assuming that $K \neq 1$. Collecting terms in (B), we have

$$(1-K)(x^2 + y^2 + c^2) - 2cx(1+K) = 0.$$

Division by $1-K$ ($\neq 0$ because $K \neq 1$) gives

$$x^2 + y^2 + c^2 - 2Lx = 0 \quad \text{where} \quad L = \frac{c(1+K)}{1-K}.$$

Completing the square in x, we finally obtain

$$(x-L)^2 + y^2 = L^2 - c^2.$$

This is a circle with center at L on the real axis and radius $\sqrt{L^2 - c^2}$.

We simplify $\sqrt{L^2 - c^2}$ as follows. First, we consider

$$\begin{aligned}
L^2 - c^2 &= \left[\frac{c(1+K)}{1-K}\right]^2 - c^2 \qquad \text{(by inserting } L\text{)} \\
&= \frac{c^2(1+K)^2}{(1-K)^2} - c^2 \\
&= c^2\left[\frac{(1+K)^2}{(1-K)^2} - 1\right] \\
&= c^2\left[\frac{(1+K)^2}{(1-K)^2} - \frac{(1-K)^2}{(1-K)^2}\right] \\
&= c^2\,\frac{1 + 2K + K^2 - (1 - 2K - K^2)}{(1-K)^2} \\
&= \frac{c^2 4K^2}{(1-K)^2}.
\end{aligned}$$

Hence

$$\sqrt{L^2 - c^2} = \sqrt{\frac{c^2 4K^2}{(1-K)^2}} = \frac{c2K}{1-K} = \frac{2ck^2}{1-k^2} \qquad \text{(using } K = k^2\text{)}.$$

Thus the radius equals $2ck^2/(1-k^2)$.

15. Potential in a sector. To solve the given problem, we note that

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

From Row 2, with Columns 3 and 6, we see that

$$x_2 = \tfrac{5}{10} = \tfrac{1}{2}.$$

Furthermore, from Row 3, with Columns 5 and 6, we obtain

$$x_4 = \frac{\frac{15}{2}}{1} = \frac{15}{2}.$$

Now x_4 appears in the second constraint, written as equation, that is,

$$2x_1 + 5x_2 + x_4 = 10.$$

Inserting $x_2 = \frac{1}{2}$ and $x_4 = \frac{15}{2}$ gives

$$2x_1 + 10 = 10, \qquad \text{hence} \qquad x_1 = 0.$$

Hence

$$\text{the minimum } -10 \text{ of } z = f(x_1, x_2) \text{ occurs at the point } \left(0, \tfrac{1}{2}\right).$$

Since this problem involves only two variables (not counting the slack variables), as a control and to better understand the problem, you may want to graph the constraints. You will notice that they determine a quadrangle. When you calculate the values of f at the four vertices of the quadrangle, you should obtain

$$0 \text{ at } (0, 0),\ 25 \text{ at } (5, 0),\ -7.5 \text{ at } (2.5, 1)\text{, and } -10 \text{ at } \left(0, \tfrac{1}{2}\right).$$

This would confirm our result.

Sec. 22.4 Simplex Method. Difficulties

Of lesser importance are two types of difficulties that are encountered with the simplex method: *degeneracy*, illustrated in Example 1 (pp. 962–965), Problem 1 and *difficulties in starting*, illustrated in Example 2 (pp. 965–967).

Problem Set 22.4. Page 968

1. Degeneracy. Choice of pivot. Undefined quotient. The given problem is

$$z = f_1(\mathbf{x}) = 7x_1 + 14x_2$$

subject to

$$0 \le x_1 \le 6,$$
$$0 \le x_2 \le 3,$$
$$7x_1 + 14x_2 \le 84.$$

6. Vertex 6 is in T and its adjacent vertices 1, 3, 5 are in S.

Since none of the six steps gave us any contradiction, we conclude that the given graph in this problem is bipartite. Take another look at the figure of our graph on p. 1005 to realize that, although the number of vertices and edges is small, the present problem is not completely trivial. We can sketch the graph in such a way that we can immediately see that it is bipartite.

17. K_4 **is planar** because we can graph it as a square A, B, C, D, then add one diagonal, say, A, C, inside, and then join B, D not by a diagonal inside (which would cross) but by a curve outside the square.

Answer to question on greedy algorithm (see p. 10 in Sec. 23.4 of this Student Solutions Manual and Study Guide). Yes, definitely, Dijkstra's algorithm is an example of a greedy algorithm, as in Steps 2 and 3 it looks for the shortest path between the current vertex and the next vertex.

Answer to self-test on Prim's and Dijkstra's algorithms (see p. 12 of Sec. 23.5). Yes, since both trees are spanning trees.