

Student Solutions Manual

Phillip E. Bedient • Richard E. Bedient

Eighth Edition

Elementary Differential Equations

Earl D. Rainville
Phillip E. Bedient
Richard E. Bedient

Student Solutions Manual

Phillip E. Bedient • Richard E. Bedient

Eighth Edition

Elementary Differential Equations

Earl D. Rainville

*Late Professor of Mathematics
University of Michigan*

Phillip E. Bedient

*Professor Emeritus of Mathematics
Franklin and Marshall College*

Richard E. Bedient

*Professor of Mathematics
Hamilton College*

PRENTICE HALL, UPPER SADDLE RIVER, NJ 07458

Assistant Editor: **Audra Walsh**
Production Editor: **Carole Suraci**
Special Projects Manager: **Barbara A. Murray**
Supplement Cover Manager: **Paul Gourhan**
Manufacturing Buyer: **Alan Fischer**

Copyright © 1997 by Prentice-Hall, Inc.
A Pearson Education Company
Upper Saddle River, NJ 07458

All rights reserved. No part of this book may be
reproduced in any form or by any means,
without permission in writing from the publisher.

Printed in the United States of America

ISBN 0-13-592783-8

Prentice-Hall International (UK) Limited, London
Prentice-Hall of Australia Pty. Limited, Sydney
Prentice-Hall Canada Inc., Toronto
Prentice-Hall Hispanoamericana, S.A., Mexico
Prentice-Hall of India Private Limited, New Delhi
Prentice-Hall of Japan, Inc., Tokyo
Pearson Education Asia Pte. Ltd., Singapore
Editora Prentice-Hall do Brasil, Ltda., Rio de Janeiro

Contents

1	Definitions; Families of Curves	1
1.2	Definitions	1
1.3	Families of Solutions	1
2	Equations of Order One	2
2.1	Separation of Variables	2
2.2	Homogeneous Functions	5
2.3	Equations with Homogeneous Coefficients	5
2.4	Exact Equations	9
2.6	The General Solution of a Linear Equation	11
	Miscellaneous Exercises	15
3	Numerical Methods	21
3.2	Euler's Method	21
3.3	A Modification of Euler's Method	22
3.4	A Method of Successive Approximation	22
3.5	An Improvement	22
3.6	The Use of Taylor's Theorem	23
3.7	The Runge-Kutta Method	24
3.8	A Continuing Method	24
4	Elementary Applications	25
4.3	Simple Chemical Conversion	25
4.4	Logistic Growth and the Price of Commodities	26
5	Additional Topics	
	on Equations of Order One	28
5.1	Integrating Factors Found by Inspection	28
5.2	The Determination of Integrating Factors	30
5.4	Bernoulli's Equation	32
5.5	Coefficients Linear in the Two Variables	34
5.6	Solutions Involving Nonelementary Integrals	35

CONTENTS

6	Linear Differential Equations	39
6.2	An Existence and Uniqueness Theorem	39
6.4	The Wronskian	39
6.8	The Fundamental Laws of Operation	40
6.9	Some Properties of Differential Operators	41
7	Linear Equations with Constant Coefficients	42
7.2	The Auxiliary Equation: Distinct Roots	42
7.3	The Auxiliary Equation: Repeated Roots	43
7.6	A Note on Hyperbolic Functions	44
	Miscellaneous Exercises	45
8	Nonhomogeneous Equations: Undetermined Coefficients	48
8.1	Construction of a Homogeneous Equation from a Specific Solution	48
8.3	The Method of Undetermined Coefficients	49
8.4	Solution by Inspection	54
9	Variation of Parameters	56
9.2	Reduction of Order	56
9.4	Solution of $y'' + y = f(x)$	58
10	Applications	70
10.3	Resonance	70
10.4	Damped Vibrations	73
10.5	The Simple Pendulum	77
11	Linear Systems of Equations	78
11.2	First-Order Systems with Constant Coefficients	78
11.4	Some Matrix Algebra	78
11.5	First-Order Systems Revisited	79
11.6	Complex Eigenvalues	81
11.7	Repeated Eigenvalues	83
11.8	The Phase Plane	85
12	Nonhomogeneous Systems of Equations	87
12.1	Nonhomogeneous Systems	87
12.2	Arms Races	88
12.4	Simple Networks	90
13	The Existence and Uniqueness of Solutions	94
13.2	An Existence and Uniqueness Theorem	94

CONTENTS

14 The Laplace Transform	95
14.3 Transforms of Elementary Functions	95
14.6 Functions of Class A	96
14.10 Periodic Functions	98
15 Inverse Transforms	101
15.1 Definition of an Inverse Transform	101
15.2 Partial Fractions	102
15.3 Initial Value Problems	102
15.4 A Step Function	108
15.5 A Convolution Theorem	110
15.6 Special Integral Equations	111
15.8 The Deflection of Beams	114
15.9 Systems of Equations	115
16 Nonlinear Equations	120
16.2 Factoring the Left Member	120
16.5 The p -Discriminant Equation	123
16.7 Clairaut's Equation	124
16.9 Independent Variable Missing	126
Miscellaneous Exercises	131
17 Power Series Solutions	134
17.5 Solutions Near an Ordinary Point	134
18 Solutions Near Regular Singular Points	143
18.1 Regular Singular Points	143
18.4 Difference of Roots Nonintegral	143
18.6 Equal Roots	152
18.7 Equal Roots, an Alternative	158
18.8 Nonlogarithmic Case	160
18.9 Logarithmic Case	165
18.10 Solution for Large x	174
18.11 Many-Term Recurrence Relations	181
Miscellaneous Exercises	185
20 Partial Differential Equations	205
20.3 Method of Separation of Variables	205
21 Orthogonal Sets of Functions	208
21.6 Other Orthogonal Sets	208
22 Fourier Series	210
22.3 Numerical Examples of Fourier Series	210
22.4 Fourier Sine Series	213
22.5 Fourier Cosine Series	215

CONTENTS

23 Boundary Value Problems	217
23.1 The One-Dimensional Heat Equation	217
23.4 Heat Conduction in a Sphere	220
23.5 The Simple Wave Equation	221
23.6 Laplace's Equation in Two Dimensions	223
24 Additional Properties of the Laplace Transform	228
24.1 Power Series and Inverse Transforms	228
24.2 The Error Function	230
24.3 Bessel Functions	231
25 Partial Differential Equations:	
Transform Methods	232
25.1 Boundary Value Problems	232
25.2 The Wave Equation	233
25.5 Diffusion in a Slab of Finite Width	233
25.6 Diffusion in a Quarter-Infinite Solid	234

Chapter 1

Definitions; Families of Curves

1.2 Definitions

All answers in this section are determined by inspection.

1. The equation is ordinary, linear in x , and of order 2.
3. The equation is ordinary, nonlinear, and of order 1.
5. The equation is ordinary, linear in y , and of order 3.
7. The equation is partial, linear in u , and of order 2.
9. The equation is ordinary, linear in x or y , and of order 2.
11. The equation is ordinary, linear in y , and of order 1.
13. The equation is ordinary, nonlinear, and of order 3.
15. The equation is ordinary, linear in y , and of order 2.

1.3 Families of Solutions

1. Rewriting the equation yields $y = \int x^3 + 2x \, dx + c$. Integrating, we have $y = \frac{1}{4}x^4 + x^2 + c$.
3. Rewriting the equation yields $y = 4 \int \cos 6x \, dx + c$. Integrating, we have $y = \frac{2}{3} \sin 6x + c$.
5. Rewriting the equation yields $y = 2 \int \frac{1}{x^2 + 2^2} \, dx + c$. Integrating, we have $y = \arctan(x/2) + c$.
7. Rewriting the equation yields $y = 3 \int e^x \, dx$. Integrating, we have $y = 3e^x + c$. Substituting the initial conditions gives $6 = 3 + c$ or $c = 3$ so $y = 3e^x + 3$.
9. As in Example 1.2, $y = ce^{4x}$. Substituting the initial conditions gives $3 = ce^0 = c$ so $y = 3e^{4x}$.
11. Rewriting the equation yields $y = 4 \int \sin 2x \, dx$. Integrating, we have $-2 \cos 2x + c$. Substituting the initial conditions gives $2 = -2 \cos \pi + c = 2 + c$ or $c = 0$ so $y = -2 \cos 2x$.

7.3. THE AUXILIARY EQUATION: REPEATED ROOTS

43

25. The auxiliary equation is $m^2 - 2m - 3 = 0$ and its roots are $m = 3, -1$. The general solution is $y = c_1 e^{3x} + c_2 e^{-x}$ and $y' = 3c_1 e^{3x} - c_2 e^{-x}$. But $y(0) = c_1 + c_2 = 4$ and $y'(0) = 3c_1 - c_2 = 0$, so that $c_1 = 1, c_2 = 3$. The particular solution is $y = e^{3x} + 3e^{-x}$. Thus $y(1) = e^3 + e^{-1}$.
27. The auxiliary equation is $m^2 - m - 6 = 0$ and its roots are $m = 3, -2$. The general solution is $y = c_1 e^{3x} + c_2 e^{-2x}$ and $y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$. But $y(0) = c_1 + c_2 = 3$ and $y'(0) = 3c_1 - 2c_2 = -1$, so that $c_1 = 1, c_2 = 2$. The particular solution is $y = e^{3x} + 2e^{-2x}$. Thus $y(1) = e^3 + 2e^{-2}$.
29. The auxiliary equation is $m^3 - 2m^2 - 5m + 6 = 0$ and its roots are $m = 1, 3, -2$. The general solution is $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$, so that $y' = c_1 e^x + 3c_2 e^{3x} - 2c_3 e^{-2x}$, and $y'' = c_1 e^x + 9c_2 e^{3x} + 4c_3 e^{-2x}$. But $y(0) = c_1 + c_2 + c_3 = 1, y'(0) = c_1 + 3c_2 - 2c_3 = -7$, and $y''(0) = c_1 + 9c_2 + 4c_3 = -1$. Thus $c_1 = 0, c_2 = -1, c_3 = 2$. The particular solution is $y = -e^{3x} + 2e^{-2x}$ and $y(1) = -e^3 + 2e^{-2}$.

7.3 The Auxiliary Equation: Repeated Roots

- The auxiliary equation is $m^2 - 6m + 9 = 0$ and its roots are $m = 3, 3$. The general solution is $y = (c_1 + c_2 x)e^{3x}$.
- The auxiliary equation is $4m^3 + 4m^2 + m = 0$ and its roots are $m = 0, -\frac{1}{2}, -\frac{1}{2}$. The general solution is $y = c_1 + (c_2 + c_3 x) \exp(-\frac{1}{2}x)$.
- The auxiliary equation is $m^4 + 6m^3 + 9m^2 = 0$ and its roots are $m = 0, 0, -3, -3$. The general solution is $y = c_1 + c_2 x + (c_3 + c_4 x)e^{-3x}$.
- The auxiliary equation is $4m^3 - 3m + 1 = 0$ and its roots are $m = -1, \frac{1}{2}, \frac{1}{2}$. The general solution is $y = (c_1 + c_2 x) \exp(\frac{1}{2}x) + c_3 e^{-x}$.
- The auxiliary equation is $m^3 + 3m^2 + 3m + 1 = 0$ and its roots are $m = -1, -1, -1$. The general solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$.
- The auxiliary equation is $m^5 - m^3 = 0$ and its roots are $m = 0, 0, 0, 1, -1$. The general solution is $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$.
- The auxiliary equation is $4m^4 + 4m^3 - 3m^2 - 2m + 1 = 0$ and its roots are $m = -1, -1, \frac{1}{2}, \frac{1}{2}$. The general solution is $y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x) \exp(\frac{1}{2}x)$.
- The auxiliary equation is $m^4 + 3m^3 - 6m^2 - 28m - 24 = 0$ and its roots are $m = -2, -2, -2, 3$. The general solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-2x} + c_4 e^{3x}$.
- The auxiliary equation is $4m^5 - 23m^3 - 33m^2 - 17m - 3 = 0$ with roots $m = -1, -1, 3, -\frac{1}{2}, -\frac{1}{2}$. The general solution is $y = (c_1 + c_2 x)e^{-x} + c_3 e^{3x} + (c_4 + c_5 x) \exp(-\frac{1}{2}x)$.
- The auxiliary equation is $m^4 - 5m^2 - 6m - 2 = 0$ and its roots are $m = 1 \pm \sqrt{3}, -1, -1$. The general solution is $y = (c_1 + c_2 x)e^{-x} + c_3 \exp[(1 + \sqrt{3})x] + c_4 \exp[(1 - \sqrt{3})x]$.
- The auxiliary equation is $m^2 + 4m + 4 = 0$ and its roots are $m = -2, -2$. The general solution is $y = (c_1 + c_2 x)e^{-2x}$ and $y' = (-2c_1 + c_2 - 2c_2 x)e^{-2x}$. But $y(0) = c_1 = 1$ and $y'(0) = -2c_1 + c_2 = -1$, so that $c_1 = c_2 = 1$. The particular solution is $y = (1 + x)e^{-2x}$.

The first of these equations can be written $\frac{dI_2}{dt} = -\frac{R_1}{L_2}I_1 - \frac{R_2}{L_2}I_2 + \frac{E}{L_2}$. Differentiating the second equation and using the third equation to eliminate I_1 yields

$$\begin{aligned} R_1 \frac{dI_1}{dt} + R_3 \frac{dI_3}{dt} + \frac{1}{C_3} I_3 &= 0, \\ R_1 \frac{dI_1}{dt} + R_3 \left(\frac{dI_1}{dt} - \frac{dI_2}{dt} \right) + \frac{1}{C_3} (I_1 - I_2) &= 0. \end{aligned}$$

Replacing $\frac{dI_2}{dt}$ by its equivalent gives us

$$\begin{aligned} (R_1 + R_3) \frac{dI_1}{dt} - \frac{R_3}{L_2} (-R_1 I_1 - R_2 I_2 + E) + \frac{1}{C_3} (I_1 - I_2) &= 0, \\ (R_1 + R_3) \frac{dI_1}{dt} = \left(-\frac{R_1 R_3}{L_2} - \frac{1}{C_3} \right) I_1 + \left(-\frac{R_2 R_3}{L_2} + \frac{1}{C_3} \right) I_2 + \frac{R_3 E}{L_2}. \end{aligned}$$

The system in I_1 and I_2 can now be written

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}' = \begin{pmatrix} -\frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} & -\frac{C_3 R_2 R_3 - L_2}{C_3 L_2 (R_1 + R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} \frac{R_3 E}{L_2 (R_1 + R_3)} \\ \frac{E}{L_2} \end{pmatrix}.$$

The nature of the solutions of this system depend upon the roots of the characteristic equation

$$\begin{vmatrix} -\frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} - m & -\frac{C_3 R_2 R_3 - L_2}{C_3 L_2 (R_1 + R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} - m \end{vmatrix} = 0,$$

which may be written

$$m^2 + \left[\frac{C_3 R_1 R_3 + L_2}{C_3 L_2 (R_1 + R_3)} + \frac{R_2}{L_2} \right] m + \frac{R_2 (C_3 R_1 R_3 + L_2)}{C_3 L_2^2 (R_1 + R_3)} - \frac{R_1 (C_3 R_2 R_3 - L_2)}{C_3 L_2^2 (R_1 + R_3)} = 0,$$

or

$$C_3 L_2 (R_1 + R_3) m^2 + [C_3 (R_1 R_2 + R_2 R_3 + R_3 R_1) + L_2] m + R_1 + R_2 = 0.$$

Note that the answer given in the book has a typographic error. The last term should be R_2 , not R_3 .

18.11. MANY-TERM RECURRENCE RELATIONS

193

The indicial equation is $(c+1)^2 = 0$ and $a_n = \frac{-a_{n-1}}{n+c+1}$ for $n \geq 1$. Solving this recurrence relation we get

$$\begin{aligned} a_n &= \frac{(-1)^n a_0}{(c+2) \cdots (c+n+1)}, \\ y_c &= x^c + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+c}}{(c+2) \cdots (c+n+1)}, \\ \frac{\partial y_c}{\partial c} &= y_c \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+c}}{(c+2) \cdots (c+n+1)} \left[-\frac{1}{c+2} - \cdots - \frac{1}{c+n+1} \right]. \end{aligned}$$

Substituting $c = -1$ gives the solutions

$$\begin{aligned} y_1 &= x^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{n!}, \\ y_2 &= y_1 \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n x^{n-1}}{n!}. \end{aligned}$$

25. Set $L(y) = x(1-2x)y'' - 2(2+x)y' + 18y$. Then

$$\begin{aligned} L(y) &= \sum_{n=0}^{\infty} [(n+c)(n+c-1) - 4(n+c)] a_n x^{n+c-1} \\ &\quad - \sum_{n=0}^{\infty} [2(n+c)(n+c-1) + 2(n+c) - 18] a_n x^{n+c} \\ &= \sum_{n=0}^{\infty} (n+c)(n+c-5) a_n x^{n+c-1} - \sum_{n=0}^{\infty} 2(n+c-3)(n+c+3) a_n x^{n+c} \\ &= c(c-5) a_0 x^{c-1} + \sum_{n=1}^{\infty} [(n+c)(n+c-5) a_n - 2(n+c-4)(n+c+2) a_{n-1}] x^{n+c-1}. \end{aligned}$$

Choosing $c = 0$ and $n(n-5)a_n = 2(n-4)(n+2)a_{n-1}$ for $n \geq 1$,

$$\begin{aligned} 1 \cdot (-4) a_1 &= 2 \cdot (-3) \cdot 3 a_0, & a_1 &= \frac{9}{2} a_0, \\ 2 \cdot (-3) a_2 &= 2 \cdot (-2) \cdot 4 a_1, & a_2 &= 12 a_0, \\ 3 \cdot (-2) a_3 &= 2 \cdot (-1) \cdot 5 a_2, & a_3 &= 20 a_0, \\ 4 \cdot (-1) a_4 &= 2 \cdot (0) \cdot 6 a_3, & a_4 &= 0, \\ 5 \cdot (0) a_5 &= 2 \cdot (1) \cdot 7 a_4, & a_5 &\text{arbitrary.} \end{aligned}$$

This linear differential equation has general solution $u(x, s) = c_1(s)e^{4sx} + c_2(s)e^{-4sx} - \frac{2}{s^2}$. In order for $\lim_{x \rightarrow \infty} u$ to exist we must take $c_1(s) = 0$. As $x \rightarrow 0$ we need to take $\frac{1}{s^2} = c_2(s) - \frac{2}{s^2}$. That is $c_2(s) = \frac{3}{s^2}$. We therefore have $u(x, s) = \frac{3}{s^2}e^{-4sx} - \frac{2}{s^2}$. An inverse transform now yields the solution $y(x, t) = 3(t - 4x)\alpha(t - 4x) - 2t$.

25.2 The Wave Equation

1. Direct application of the Laplace transform gives us the transformed system

$$s^2u - s(x - x^2) = \frac{d^2u}{dx^2}, \quad x \rightarrow 0^+, u \rightarrow 0, \quad x \rightarrow 1^-, u \rightarrow 0.$$

The linear equation has as its general solution $u = \frac{x}{s} - \frac{x^2}{s} - \frac{2}{s^3} + c_1e^{-sx} + c_2e^{sx}$. The condition $x \rightarrow 0^+, u \rightarrow 0$ implies that $c_1 + c_2 = 2/s^3$. The condition $x \rightarrow 1^-, u \rightarrow 0$ implies that $e^{-s}c_1 + e^sc_2 = 2/s^3$. From these two equations we obtain

$$c_1 = \frac{2}{s^3(1 + e^{-s})} = \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^n e^{-ns} \quad \text{and} \quad c_2 = \frac{2e^{-s}}{s^3(1 + e^{-s})} = \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^{n+1} e^{-(n+1)s}.$$

Therefore

$$u(x, s) = \frac{x}{s} - \frac{x^2}{s} - \frac{2}{s^3} + \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^n e^{-(n+x)s} + \frac{2}{s^3} \sum_{n=0}^{\infty} (-1)^{n+1} e^{-(n+1-x)s}.$$

The inverse transform now yields

$$y(x, t) = x - x^2 - t^2 + \sum_{n=0}^{\infty} (-1)^n [(t - n - x)^2 \alpha(t - n - x) + (t - n - 1 + x)^2 \alpha(t - n - 1 + x)].$$

25.5 Diffusion in a Slab of Finite Width

1. Direct application of the Laplace transform gives us the transformed system

$$sw - 1 = \frac{d^2w}{dx^2}, \quad x \rightarrow 0^+, w \rightarrow 0, \quad x \rightarrow 1^-, \frac{dw}{dx} \rightarrow 0.$$

The linear equation has as its general solution $w = \frac{1}{s} + c_1 \sinh(x\sqrt{s}) + c_2 \cosh(x\sqrt{s})$. The condition $x \rightarrow 0^+, w \rightarrow 0$ implies that $c_2 = -1/s$. Thus

$$\frac{dw}{dx} = \sqrt{s}c_1 \cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}} \sinh(x\sqrt{s}).$$