Student Solutions Manual

Phillip E. Bedient • Richard E. Bedient

Eighth Edition

Elementary Differential Equations

Earl D. Rainville
Phillip E. Bedient
Richard E. Bedient

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Earl D. Rainville
Late Professor of Mathematics
University of Michigan

Phillip E. Bedient
Professor Emeritus of Mathematics
Franklin and Marshall College

Richard E. Bedient
Professor of Mathematics
Hamilton College

PRENTICE HALL, UPPER SADDLE RIVER, NJ 07458

Assistant Editor: ***Audra Walsh***
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Chapter 1

Definitions; Families of Curves

1.2 Definitions

All answers in this section are determined by inspection.

1. The equation is ordinary, linear in x, and of order 2.
3. The equation is ordinary, nonlinear, and of order 1.
5. The equation is ordinary, linear in y, and of order 3.
7. The equation is partial, linear in u, and of order 2.
9. The equation is ordinary, linear in x or y, and of order 2.
11. The equation is ordinary, linear in y, and of order 1.
13. The equation is ordinary, nonlinear, and of order 3.
15. The equation is ordinary, linear in y, and of order 2.

1.3 Families of Solutions

1. Rewriting the equation yields $y = \int x^3 + 2x\,dx + c$. Integrating, we have $y = \frac{1}{4}x^4 + x^2 + c$.
3. Rewriting the equation yields $y = 4\int \cos 6x\,dx + c$. Integrating, we have $y = \frac{2}{3}\sin 6x + c$.
5. Rewriting the equation yields $y = 2\int \frac{1}{x^2+2^2}\,dx + c$. Integrating, we have $y = \arctan(x/2) + c$.
7. Rewriting the equation yields $y = 3\int e^x\,dx$. Integrating, we have $y = 3e^x + c$. Substituting the initial conditions gives $6 = 3 + c$ or $c = 3$ so $y = 3e^x + 3$.
9. As in Example 1.2, $y = ce^{4x}$. Substituting the initial conditions gives $3 = ce^0 = c$ so $y = 3e^{4x}$.
11. Rewriting the equation yields $y = 4\int \sin 2x\,dx$. Integrating, we have $-2\cos 2x + c$. Substituting the initial conditions gives $2 = -2\cos\pi + c = 2 + c$ or $c = 0$ so $y = -2\cos 2x$.

25. The auxiliary equation is $m^2 - 2m - 3 = 0$ and its roots are $m = 3,\ -1$. The general solution is $y = c_1 e^{3x} + c_2 e^{-x}$ and $y' = 3c_1 e^{3x} - c_2 e^{-x}$. But $y(0) = c_1 + c_2 = 4$ and $y'(0) = 3c_1 - c_2 = 0$, so that $c_1 = 1,\ c_2 = 3$. The particular solution is $y = e^{3x} + 3e^{-x}$. Thus $y(1) = e^3 + e^{-1}$.

27. The auxiliary equation is $m^2 - m - 6 = 0$ and its roots are $m = 3,\ -2$. The general solution is $y = c_1 e^{3x} + c_2 e^{-2x}$ and $y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$. But $y(0) = c_1 + c_2 = 3$ and $y'(0) = 3c_1 - 2c_2 = -1$, so that $c_1 = 1,\ c_2 = 2$. The particular solution is $y = e^{3x} + 2e^{-2x}$. Thus $y(1) = e^3 + 2e^{-2}$.

29. The auxiliary equation is $m^3 - 2m^2 - 5m + 6 = 0$ and its roots are $m = 1,\ 3,\ -2$. The general solution is $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$, so that $y' = c_1 e^x + 3c_2 e^{3x} - 2c_3 e^{-2x}$, and $y'' = c_1 e^x + 9c_2 e^{3x} + 4c_3 e^{-2x}$. But $y(0) = c_1 + c_2 + c_3 = 1$, $y'(0) = c_1 + 3c_2 - 2c_3 = -7$, and $y''(0) = c_1 + 9c_2 + 4c_3 = -1$. Thus $c_1 = 0,\ c_2 = -1,\ c_3 = 2$. The particular solution is $y = -e^{3x} + 2e^{-2x}$ and $y(1) = -e^3 + 2e^{-2}$.

7.3 The Auxiliary Equation: Repeated Roots

1. The auxiliary equation is $m^2 - 6m + 9 = 0$ and its roots are $m = 3,\ 3$. The general solution is $y = (c_1 + c_2 x)e^{3x}$.

3. The auxiliary equation is $4m^3 + 4m^2 + m = 0$ and its roots are $m = 0,\ -\frac{1}{2},\ -\frac{1}{2}$. The general solution is $y = c_1 + (c_2 + c_3 x)\exp\left(-\frac{1}{2}x\right)$.

5. The auxiliary equation is $m^4 + 6m^3 + 9m^2 = 0$ and its roots are $m = 0,\ 0,\ -3,\ -3$. The general solution is $y = c_1 + c_2 x + (c_3 + c_4 x)e^{-3x}$.

7. The auxiliary equation is $4m^3 - 3m + 1 = 0$ and its roots are $m = -1,\ \frac{1}{2},\ \frac{1}{2}$. The general solution is $y = (c_1 + c_2 x)\exp\left(\frac{1}{2}x\right) + c_3 e^{-x}$.

9. The auxiliary equation is $m^3 + 3m^2 + 3m + 1 = 0$ and its roots are $m = -1,\ -1,\ -1$. The general solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$.

11. The auxiliary equation is $m^5 - m^3 = 0$ and its roots are $m = 0,\ 0,\ 0,\ 1,\ -1$. The general solution is $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$.

13. The auxiliary equation is $4m^4 + 4m^3 - 3m^2 - 2m + 1 = 0$ and its roots are $m = -1,\ -1,\ \frac{1}{2},\ \frac{1}{2}$. The general solution is $y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)\exp\left(\frac{1}{2}x\right)$.

15. The auxiliary equation is $m^4 + 3m^3 - 6m^2 - 28m - 24 = 0$ and its roots are $m = -2,\ -2,\ -2,\ 3$. The general solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-2x} + c_4 e^{3x}$.

17. The auxiliary equation is $4m^5 - 23m^3 - 33m^2 - 17m - 3 = 0$ with roots $m = -1,\ -1,\ 3,\ -\frac{1}{2},\ -\frac{1}{2}$. The general solution is $y = (c_1 + c_2 x)e^{-x} + c_3 e^{3x} + (c_4 + c_5 x)\exp\left(-\frac{1}{2}x\right)$.

19. The auxiliary equation is $m^4 - 5m^2 - 6m - 2 = 0$ and its roots are $m = 1 \pm \sqrt{3},\ -1,\ -1$. The general solution is $y = (c_1 + c_2 x)e^{-x} + c_3 \exp\left[(1 + \sqrt{3})x\right] + c_4 \exp\left[(1 - \sqrt{3})x\right]$.

21. The auxiliary equation is $m^2 + 4m + 4 = 0$ and its roots are $m = -2,\ -2$. The general solution is $y = (c_1 + c_2 x)e^{-2x}$ and $y' = (-2c_1 + c_2 - 2c_2 x)e^{-2x}$. But $y(0) = c_1 = 1$ and $y'(0) = -2c_1 + c_2 = -1$, so that $c_1 = c_2 = 1$. The particular solution is $y = (1 + x)e^{-2x}$.

The first of these equations can be written $\frac{dI_2}{dt} = -\frac{R_1}{L_2}I_1 - \frac{R_2}{L_2}I_2 + \frac{E}{L_2}$. Differentiating the second equation and using the third equation to eliminate I_1 yields

$$R_1\frac{dI_1}{dt} + R_3\frac{dI_3}{dt} + \frac{1}{C_3}I_3 = 0,$$
$$R_1\frac{dI_1}{dt} + R_3\left(\frac{dI_1}{dt} - \frac{dI_2}{dt}\right) + \frac{1}{C_3}(I_1 - I_2) = 0.$$

Replacing $\frac{dI_2}{dt}$ by its equivalent gives us

$$(R_1 + R_3)\frac{dI_1}{dt} - \frac{R_3}{L_2}(-R_1I_1 - R_2I_2 + E) + \frac{1}{C_3}(I_1 - I_2) = 0,$$
$$(R_1 + R_3)\frac{dI_1}{dt} = \left(-\frac{R_1R_3}{L_2} - \frac{1}{C_3}\right)I_1 + \left(-\frac{R_2R_3}{L_2} + \frac{1}{C_3}\right)I_2 + \frac{R_3E}{L_2}.$$

The system in I_1 and I_2 can now be written

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}' = \begin{pmatrix} -\frac{C_3R_1R_3+L_2}{C_3L_2(R_1+R_3)} & -\frac{C_3R_2R_3-L_2}{C_3L_2(R_1+R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} \frac{R_3E}{L_2(R_1+R_3)} \\ \frac{E}{L_2} \end{pmatrix}.$$

The nature of the solutions of this system depend upon the roots of the characteristic equation

$$\begin{vmatrix} -\frac{C_3R_1R_3+L_2}{C_3L_2(R_1+R_3)} - m & -\frac{C_3R_2R_3-L_2}{C_3L_2(R_1+R_3)} \\ -\frac{R_1}{L_2} & -\frac{R_2}{L_2} - m \end{vmatrix} = 0,$$

which may be written

$$m^2 + \left[\frac{C_3R_1R_3 + L_2}{C_3L_2(R_1 + R_3)} + \frac{R_2}{L_2}\right]m + \frac{R_2(C_3R_1R_3 + L_2)}{C_3L_2^2(R_1 + R_3)} - \frac{R_1(C_3R_2R_3 - L_2)}{C_3L_2^2(R_1 + R_3)} = 0,$$

or

$$C_3L_2(R_1 + R_3)m^2 + \left[C_3(R_1R_2 + R_2R_3 + R_3R_1) + L_2\right]m + R_1 + R_2 = 0.$$

Note that the answer given in the book has a typographic error. The last term should be R_2, not R_3.

The indicial equation is $(c+1)^2 = 0$ and $a_n = \dfrac{-\,a_{n-1}}{n+c+1}$ for $n \geq 1$. Solving this recurrence relation we get

$$\begin{aligned}
a_n &= \frac{(-1)^n\, a_0}{(c+2)\cdots(c+n+1)},\\
y_c &= x^c + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{n+c}}{(c+2)\cdots(c+n+1)},\\
\frac{\partial y_c}{\partial c} &= y_c \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{n+c}}{(c+2)\cdots(c+n+1)} \left[-\frac{1}{c+2} - \cdots - \frac{1}{c+n+1}\right].
\end{aligned}$$

Substituting $c = -1$ gives the solutions

$$\begin{aligned}
y_1 &= x^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{n-1}}{n!},\\
y_2 &= y_1 \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n\, x^{n-1}}{n!}.
\end{aligned}$$

25. Set $L(y) = x(1-2x)y'' - 2(2+x)y' + 18y$. Then

$$\begin{aligned}
L(y) &= \sum_{n=0}^{\infty} \big[(n+c)(n+c-1) - 4(n+c)\big] a_n x^{n+c-1}\\
&\quad - \sum_{n=0}^{\infty} \big[2(n+c)(n+c-1) + 2(n+c) - 18\big] a_n x^{n+c}\\
&= \sum_{n=0}^{\infty} (n+c)(n+c-5) a_n x^{n+c-1} - \sum_{n=0}^{\infty} 2(n+c-3)(n+c+3) a_n x^{n+c}\\
&= c(c-5) a_0 x^{c-1} + \sum_{n=1}^{\infty} \big[(n+c)(n+c-5)a_n - 2(n+c-4)(n+c+2)a_{n-1}\big] x^{n+c-1}.
\end{aligned}$$

Choosing $c = 0$ and $n(n-5)a_n = 2(n-4)(n+2)a_{n-1}$ for ≥ 1,

$$\begin{aligned}
1 \cdot (-4)\, a_1 &= 2 \cdot (-3) \cdot 3\, a_0, && a_1 = \tfrac{9}{2}\, a_0,\\
2 \cdot (-3)\, a_2 &= 2 \cdot (-2) \cdot 4\, a_1, && a_2 = 12\, a_0,\\
3 \cdot (-2)\, a_3 &= 2 \cdot (-1) \cdot 5\, a_2, && a_3 = 20\, a_0,\\
4 \cdot (-1)\, a_4 &= 2 \cdot (0) \cdot 6\, a_3, && a_4 = 0,\\
5 \cdot (0)\, a_5 &= 2 \cdot (1) \cdot 7\, a_4, && a_5 \text{ arbitrary}.
\end{aligned}$$

This linear differential equation has general solution $u(x,\,s) = c_1(s)e^{4sx} + c_2(s)e^{-4sx} - \dfrac{2}{s^2}$. In order for $\lim_{x\to\infty}$ to exist we must take $c_1(s) = 0$. As $x \to 0$ we need to take $\dfrac{1}{s^2} = c_2(s) - \dfrac{2}{s^2}$. That is $c_2(s) = \dfrac{3}{s^2}$. We therefore have $u(x,\,s) = \dfrac{3}{s^2}e^{-4sx} - \dfrac{2}{s^2}$. An inverse transform now yields the solution $y(x,\,t) = 3(t-4x)\alpha(t-4x) - 2t$.

25.2 The Wave Equation

1. Direct application of the Laplace transform gives us the transformed system

$$s^2u - s(x-x^2) = \frac{d^2u}{dx^2}, \quad x \to 0^+,\ u \to 0,\ x \to 1^-,\ u \to 0.$$

The linear equation has as its general solution $u = \dfrac{x}{s} - \dfrac{x^2}{s} - \dfrac{2}{s^3} + c_1e^{-sx} + c_2e^{sx}$. The condition $x \to 0^+,\ u \to 0$ implies that $c_1 + c_2 = 2/s^3$. The condition $x \to 1^-,\ u \to 0$ implies that $e^{-s}c_1 + e^{s}c_2 = 2/s^3$. From these two equations we obtain

$$c_1 = \frac{2}{s^3(1+e^{-s})} = \frac{2}{s^3}\sum_{n=0}^{\infty}(-1)^n e^{-ns} \quad \text{and} \quad c_2 = \frac{2e^{-s}}{s^3(1+e^{-s})} = \frac{2}{s^3}\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)s}.$$

Therefore

$$u(x,\,s) = \frac{x}{s} - \frac{x^2}{s} - \frac{2}{s^3} + \frac{2}{s^3}\sum_{n=0}^{\infty}(-1)^n e^{-(n+x)s} + \frac{2}{s^3}\sum_{n=0}^{\infty}(-1)^n e^{-(n+1-x)s}.$$

The inverse transform now yields

$$y(x,\,t) = x - x^2 - t^2 + \sum_{n=0}^{\infty}(-1)^n\big[(t-n-x)^2\alpha(t-n-x) + (t-n-1+x)^2\alpha(t-n-1+x)\big].$$

25.5 Diffusion in a Slab of Finite Width

1. Direct application of the Laplace transform gives us the transformed system

$$sw - 1 = \frac{d^2w}{dx^2}, \quad x \to 0^+,\ w \to 0,\ x \to 1^-,\ \frac{dw}{dx} \to 0.$$

The linear equation has as its general solution $w = \dfrac{1}{s} + c_1\sinh(x\sqrt{s}) + c_2\cosh(x\sqrt{s})$. The condition $x \to 0^+,\ w \to 0$ implies that $c_2 = -1/s$. Thus

$$\frac{dw}{dx} = \sqrt{s}c_1\cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}}\sinh(x\sqrt{s}).$$