Chapter 1

Functions

1.1 The Real Numbers

1. Notice that $4 \cdot 16 = 64 > 63 = 9 \cdot 7$. Multiplying by the positive number $1/(9 \cdot 16)$, we have $(4 \cdot 16)/(9 \cdot 16) > (9 \cdot 7)/(9 \cdot 16)$ or $\frac{4}{9} > \frac{7}{16}$. Therefore, $a > b$.

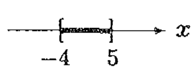
2. Notice that $7 \cdot (0.142857) = 0.999999 < 1$. Multiplying by the *negative* number $-\frac{1}{7}$, we have that $-7 \cdot (0.142857)/7 > -\frac{1}{7}$ or $-0.142857 > -\frac{1}{7}$. Therefore $a < b$.

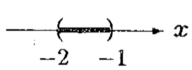
3. Since $\pi^2 > (3.14)^2 = 9.8596$, we have $a > b$.

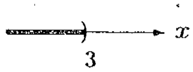
4. Since $(3.2)^2 = 10.24$, $a > b$.

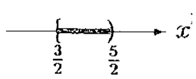
5. $(1.41)^2 = 1.9881 < 2$, so $\sqrt{2} > 1.41$.

6. $(3.3)^2 = 10.89 < 11$, so $\sqrt{11} > 3.3$.

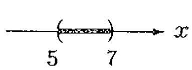
7. Closed, bounded: 

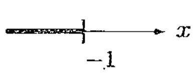
8. Open, bounded: 

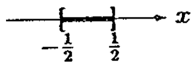
9. Open, unbounded: 

10. Half-open, bounded: 

11. Closed, unbounded: 

12. Open, bounded 

13. Closed, unbounded: 

14. Closed, bounded: 

15. $(-3,4)$

16. $(-\infty,3)$

17. $(1,\infty)$

18. $(-\infty,\infty)$

19. If $-6x-2>5$, then $-6x>7$, so $x<-\frac{7}{6}$. Thus the solution is $(-\infty,-\frac{7}{6})$.

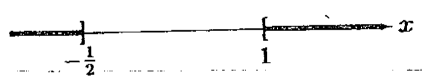
20. If $4-3x\geq 7$, then $-3x\geq 3$, so $x\leq -1$. Thus the solution is $(-\infty,-1]$.

21. If $-1\leq 2x-3<4$, then $2\leq 2x<7$, so $1\leq x<\frac{7}{2}$. Thus the solution is $[1,\frac{7}{2})$.

22. If $-0.1<3x+4<0.1$, then $-4.1<3x<-3.9$, so $-4.1/3<x<-1.3$. Thus the solution is $(-4.1/3,-1.3)$.

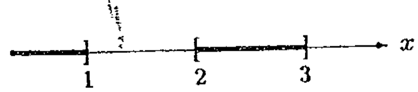
23. From the diagram we see that the solution is the union of $(-\infty,-\frac{1}{2}]$ and $[1,\infty)$.

$$\begin{array}{r l}
x-1 & - - - - - - - 0 + + + \\
x+\frac{1}{2} & - - 0 + + + + + + + + \\
(x-1)(x+\frac{1}{2}) & + + 0 - - - - 0 + + +
\end{array}$$



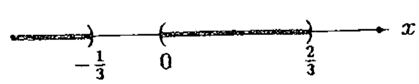
24. From the diagram we see that the solution is the union of $(-\infty,1]$ and $[2,3]$.

$$\begin{array}{r l}
x-1 & - - 0 + + + + + + + + \\
x-2 & - - - - - 0 + + + + + \\
x-3 & - - - - - - - - 0 + + \\
(x-1)(x-2)(x-3) & - - 0 + + 0 - - 0 + +
\end{array}$$



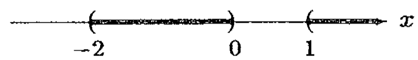
25. From the diagram we see that the solution is the union of $(-\infty,-\frac{1}{3})$ and $(0,\frac{2}{3})$.

$$\begin{array}{r l}
x & - - - - 0 + + + + + + \\
x-\frac{2}{3} & - - - - - - - - 0 + + \\
x+\frac{1}{3} & - - 0 + + + + + + + + \\
x(x-\frac{2}{3})(x+\frac{1}{3}) & - - 0 + 0 - - - 0 + +
\end{array}$$



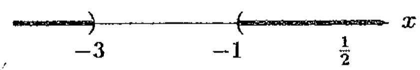
26. From the diagram we see that the solution is the union of $(-2, 0)$ and $(1, \infty)$.

$$\begin{array}{r l}
x & - - - - - - 0 + + + + \\
x-1 & - - - - - - - - 0 + + \\
x+2 & - - 0 + + + + + + + + \\
\dfrac{x}{(x-1)(x+2)} & - - \quad + + + 0 - \quad + +
\end{array}$$



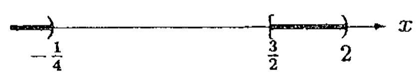
27. From the diagram we see that the solution is the union of $(-\infty, -3)$ and $(-1, \infty)$.

$$\begin{array}{r l}
(2x-1)^2 & + + + + + + + + + 0 + \\
x+1 & - - - - - - 0 + + + + \\
x+3 & - - 0 + + + + + + + + \\
\dfrac{(2x-1)^2}{(x+1)(x+3)} & + + \quad - - - \quad + + 0 +
\end{array}$$



28. From the diagram we see that the solution is the union of $(-\infty, -\frac{1}{4}]$ and $[\frac{3}{2}, 2)$.

$$\begin{array}{r l}
2x-3 & - - - - - - - - 0 + + + \\
4x+1 & - 0 + + + + + + + + + \\
x-2 & - - - - - - - - - - 0 + \\
\dfrac{(2x-3)(4x+1)}{x-2} & - 0 + + + + + 0 - \quad +
\end{array}$$



29. The given inequality is equivalent to $2x^2(2x-3) \leq 0$. From the diagram we see that the solution is $(-\infty, \frac{3}{2}]$.

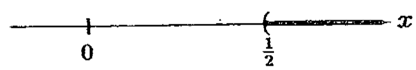
$$\begin{array}{r l}
x^2 & + + 0 + + + + + + + + \\
2x-3 & - - - - - - - 0 + + + \\
2x^2(2x-3) & - - 0 - - - - 0 + + +
\end{array}$$

30. The given inequality is equivalent to $(3x+1)(x-1) \geq 0$. From the diagram we see that the solution is the union of $(-\infty, -\frac{1}{3}]$ and $[1, \infty)$.

$3x+1$	− − 0 + + + + + + + +
$x-1$	− − − − − − − 0 + + +
$(3x+1)(x-1)$	+ + 0 − − − − 0 + + +

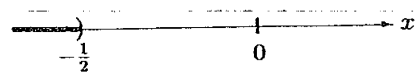
31. The given inequality is equivalent to $(8x^3-1)/x^2>0$. From the diagram we see that the solution is $(\frac{1}{2},\infty)$.

$8x^3-1$	− − − − − − − 0 + + +
x^2	+ + 0 + + + + + + + +
$\dfrac{8x^3-1}{x^2}$	− − − − − − 0 + + +



32. The given inequality is equivalent to $(8x^3+1)/x^2<0$. From the diagram we see that the solution is $(-\infty,-\frac{1}{2})$.

$8x^3+1$	− − 0 + + + + + + + +
x^2	+ + + + + + + 0 + + +
$\dfrac{8x^3+1}{x^2}$	− − 0 + + + + + + +

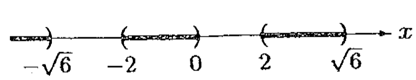


33. The given inequality is equivalent to

$$\frac{4x(x+\sqrt{6})(x-\sqrt{6})}{(x+2)(x-2)}<0.$$

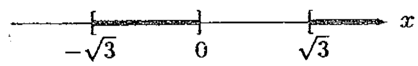
From the diagram we see that the solution is the union of $(-\infty,-\sqrt{6})$, $(-2,0)$, and $(2,\sqrt{6})$.

x	− − − − − 0 + + + + +
$x+\sqrt{6}$	− 0 + + + + + + + + +
$x-\sqrt{6}$	− − − − − − − − − 0 +
$x+2$	− − − 0 + + + + + + +
$x-2$	− − − − − − − 0 + + +
$\dfrac{4x(x+\sqrt{6})(x-\sqrt{6})}{(x+2)(x-2)}$	− 0 + − 0 + − 0 +



34. Since $(x^2+1)^3 > 0$ for all x, the given inequality is equivalent to $2x(x+\sqrt{3})(x-\sqrt{3}) \geq 0$. From the diagram we see that the solution is the union of $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$.

$$\begin{array}{r l}
x & - \; - \; - \; - \; - \; 0 \; + \; + \; + \; + \; + \\
x+\sqrt{3} & - \; - \; 0 \; + \; + \; + \; + \; + \; + \; + \; + \\
x-\sqrt{3} & - \; - \; - \; - \; - \; - \; - \; - \; 0 \; + \; + \\
2x(x+\sqrt{3})(x+\sqrt{3}) & - \; - \; 0 \; + \; + \; 0 \; - \; - \; 0 \; + \; +
\end{array}$$

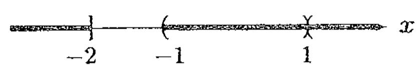


35. The given inequality is equivalent to

$$\frac{(t+2)(t-1)}{(t+1)^3(t-1)^3} \geq 0 \quad \text{or} \quad \frac{t+2}{(t+1)^3(t-1)^2} \geq 0.$$

From the diagram we see that the solution is the union of $(-\infty, -2]$, $(-1, 1)$, and $(1, \infty)$.

$$\begin{array}{r l}
t+2 & - \; - \; 0 \; + \; + \; + \; + \; + \; + \; + \; + \\
(t+1)^3 & - \; - \; - \; - \; 0 \; + \; + \; + \; + \; + \; + \\
(t-1)^2 & + \; + \; + \; + \; + \; + \; + \; + \; 0 \; + \; + \\
\dfrac{t+2}{(t+1)^3(t-1)^2} & + \; + \; 0 \; - \quad + \; + \; + \quad + \; +
\end{array}$$

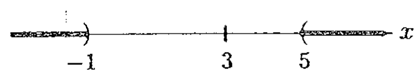


36. The given inequality is equivalent to

$$\frac{(t+1)(t-3)}{(t-5)(t-3)} > 0,$$

that is, $(t+1)/(t-5) > 0$ and $t \neq 3$. From the diagram we see that the solution is the union of $(-\infty, -1)$ and $(5, \infty)$.

$$\begin{array}{r l}
t+1 & - \; - \; 0 \; + \; + \; + \; + \; + \; + \; + \; + \\
t-5 & - \; - \; - \; - \; - \; - \; - \; - \; 0 \; + \; + \\
\dfrac{(t+1)(t-3)}{(t-5)(t-3)} & + \; + \; 0 \; - \; - \; - \quad - \quad + \; +
\end{array}$$



37. Observe that $\sqrt{9-6x}$ is defined only for $x \leq \frac{9}{6} = \frac{3}{2}$ and that $\sqrt{9-6x} > 0$ for $x < \frac{3}{2}$. Thus the given inequality is equivalent to the pair of inequalities $x < \frac{3}{2}$ and $2 - x > 0$ (that is, $x < 2$). Thus the solution is $(-\infty, \frac{3}{2})$.

38. Observe that $(1-x^2)^{1/2}$ is defined only for $-1 \leq x \leq 1$ and that $(1-x^2)^{1/2} > 0$ for $-1 < x < 1$. Moreover, $2x^2 - 1 = 2(x+\sqrt{2}/2)(x-\sqrt{2}/2)$. Thus the given inequality is equivalent to the pair of

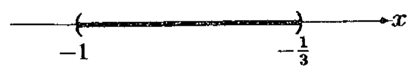
inequalities $-1 < x < 1$ and $2(x + \sqrt{2}/2)(x - \sqrt{2}/2) < 0$ (that is, $-\sqrt{2}/2 < x < \sqrt{2}/2$). Thus the solution is $(-\sqrt{2}/2, \sqrt{2}/2)$.

39. The given inequality is equivalent to

$$\frac{1}{x+1} - \frac{3}{2} > 0 \quad \text{or} \quad \frac{-3(x+\frac{1}{3})}{2(x+1)} > 0.$$

From the diagram we see that the solution is $(-1, -\frac{1}{3})$.

$$\begin{array}{rccccccccccc}
x+\frac{1}{3} & - & - & - & - & - & - & - & - & 0 & + & + \\
x+1 & - & - & 0 & + & + & + & + & + & + & + & + \\
\dfrac{-3\left(x+\frac{1}{3}\right)}{2(x+1)} & - & - & & + & + & + & + & + & 0 & - & -
\end{array}$$

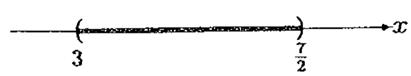


40. The given inequality is equivalent to

$$\frac{1}{3-x} + 2 < 0, \quad \text{or} \quad \frac{-2(x-\frac{7}{2})}{3-x} < 0.$$

From the diagram we see that the solution is $(3, \frac{7}{2})$.

$$\begin{array}{rccccccccccc}
x-\frac{7}{2} & - & - & - & - & - & - & - & - & 0 & + & + \\
3-x & + & + & 0 & - & - & - & - & - & - & - & - \\
\dfrac{-2\left(x-\frac{7}{2}\right)}{3-x} & + & + & & - & - & - & - & - & 0 & + & +
\end{array}$$



41. The given inequality is equivalent to

$$\frac{x+1}{x-1} - \frac{1}{2} \le 0, \quad \text{or} \quad \frac{x+3}{2(x-1)} \le 0.$$

From the diagram we see that the solution is $[-3, 1)$.

$$\begin{array}{rccccccccccc}
x+3 & - & - & 0 & + & + & + & + & + & + & + & + \\
x-1 & - & - & - & - & - & - & - & - & 0 & + & + \\
\dfrac{x+3}{2(x-1)} & + & + & 0 & - & - & - & - & - & & + & +
\end{array}$$

42. The given inequality is equivalent to

$$\frac{2-5x}{3-4x}+2\geq 0 \quad \text{or} \quad \frac{-13x+8}{3-4x}\geq 0 \quad \text{or} \quad \frac{13(x-8/13)}{4(x-3/4)}\geq 0.$$

From the diagram we see that the solution is the union of $(-\infty, \frac{8}{13}]$ and $(\frac{3}{4}, \infty)$.

$$\begin{array}{rl}
x-\frac{8}{13} & -\ -\ 0\ +\ +\ +\ +\ +\ +\ +\ + \\
x-\frac{3}{4} & -\ -\ -\ -\ -\ -\ -\ -\ -\ 0\ +\ + \\
\dfrac{13\left(x-\frac{8}{13}\right)}{4\left(x-\frac{3}{4}\right)} & +\ +\ 0\ -\ -\ -\ -\ -\ \quad +\ +
\end{array}$$

43. $-|-3|=-3$

44. $\left|-\sqrt{2}\right|^2=(\sqrt{2})^2=2$

45. $|-5|+|5|=5+5=10$

46. $|-5|-|5|=5-5=0$

47. $|x|=1$ if $x=1$ or $-x=1$; the solution is -1, 1.

48. $|x|=\pi$ if $x=\pi$ or $-x=\pi$; the solution is $-\pi$, π.

49. $|x-1|=2$ if $x-1=2$ (so that $x=3$), or $-(x-1)=2$ (so that $-x+1=2$, or $x=-1$); the solution is -1, 3.

50. $|2x-\frac{1}{2}|=\frac{1}{2}$ if $2x-\frac{1}{2}=\frac{1}{2}$ (so that $2x=1$, or $x=\frac{1}{2}$), or $-(2x-\frac{1}{2})=\frac{1}{2}$ (so that $-2x=0$, or $x=0$); the solution is 0, $\frac{1}{2}$.

51. $|6x+5|=0$ if $6x+5=0$, or $x=-\frac{5}{6}$; the solution is $-\frac{5}{6}$.

52. $|3-4x|=2$ if $3-4x=2$ (so that $-4x=-1$, or $x=\frac{1}{4}$), or $-(3-4x)=2$ (so that $4x=5$, or $x=\frac{5}{4}$); the solution is $\frac{1}{4}$, $\frac{5}{4}$.

53. If $|x|=|x|^2$, then either $|x|=0$ or we may divide by $|x|$ to obtain $1=|x|$ (so that $x=-1$ or $x=1$). The solution is -1, 0, 1.

54. If $|x|=|1-x|$, then either $x=1-x$ or $-x=1-x$. If $x=1-x$, then $2x=1$, or $x=\frac{1}{2}$. If $-x=1-x$, then $0=1$, which is impossible. The solution is $\frac{1}{2}$.

55. If $|x+1|^2+3|x+1|-4=0$, then $(|x+1|+4)(|x+1|-1)=0$. Since $|x+1|+4\neq 0$ it follows that $|x+1|-1=0$, or $|x+1|=1$. Thus either $x+1=1$ (so that $x=0$), or $-(x+1)=1$ (so that $-x=2$, or $x=-2$). The solution is 0, -2.

56. Let $u = |x-2|$. Then $|x-2|^2 - |x-2| = 6$ becomes $u^2 - u = 6$, or $u^2 - u - 6 = 0$, so that $(u-3)(u+2) = 0$, and thus $u = -2$ or $u = 3$. Since $u = |x-2| \geq 0$, it follows that $u = 3$, so $3 = u = |x-2|$. Therefore either $x - 2 = 3$ (so that $x = 5$), or $-(x-2) = 3$ (so that $x = -1$). The solution is -1, 5.

57. If $|x+4| = |x-4|$, then either $x+4 = x-4$ (so that $4 = -4$, which is impossible), or $x+4 = -(x-4)$ (so that $2x = 0$, or $x = 0$). The solution is 0.

58. If $|x-1| = |2x+1|$, then either $x - 1 = 2x + 1$ (so that $x = -2$), or $x - 1 = -(2x+1)$ (so that $3x = 0$, or $x = 0$). The solution is -2, 0.

59. If $|x-2| < 1$, then $-1 < x - 2 < 1$, or $1 < x < 3$. The solution is $(1, 3)$.

60. If $|x-4| < 0.1$, then $-0.1 < x - 4 < 0.1$, or $3.9 < x < 4.1$. The solution is $(3.9, 4.1)$.

61. If $|x+1| < 0.01$, then $-0.01 < x + 1 < 0.01$, or $-1.01 < x < -0.99$. The solution is $(-1.01, -0.99)$.

62. If $|x + \frac{1}{2}| \leq 2$, then $-2 \leq x + \frac{1}{2} \leq 2$, or $-\frac{5}{2} \leq x \leq \frac{3}{2}$. The solution is $[-\frac{5}{2}, \frac{3}{2}]$.

63. If $|x+3| \geq 3$, then $x + 3 \geq 3$ (so that $x \geq 0$), or $-(x+3) \geq 3$ (so that $x \leq -6$). The solution is the union of $(-\infty, -6]$ and $[0, \infty)$.

64. If $|x - 0.3| > 1.5$, then $x - 0.3 > 1.5$ (so that $x > 1.8$) or $-(x - 0.3) > 1.5$ (so that $-x > 1.2$, or $x < -1.2$). The solution is the union of $(-\infty, -1.2)$ and $(1.8, \infty)$.

65. If $|2x+1| \geq 1$, then either $2x + 1 \geq 1$ (so that $x \geq 0$), or $-(2x+1) \geq 1$, (so that $2x \leq -2$, or $x \leq -1$). The solution is the union of $(-\infty, -1]$ and $[0, \infty)$.

66. If $|3x - 5| \leq 2$, then $-2 \leq 3x - 5 \leq 2$, so $3 \leq 3x \leq 7$, or $1 \leq x \leq \frac{7}{3}$. The solution is $[1, \frac{7}{3}]$.

67. If $|2x - \frac{1}{3}| > \frac{2}{3}$, then either $2x - \frac{1}{3} > \frac{2}{3}$ (so that $2x > 1$, or $x > \frac{1}{2}$), or $-(2x - \frac{1}{3}) > \frac{2}{3}$ (so that $-2x > \frac{1}{3}$, or $x < -\frac{1}{6}$). The solution is the union of $(-\infty, -\frac{1}{6})$ and $(\frac{1}{2}, \infty)$.

68. If $0 < |x-1| < 0.5$, then $0 < |x-1|$ (so that $x \neq 1$) and $|x-1| < 0.5$ (so that $1 - 0.5 < x < 1 + 0.5$, or $0.5 < x < 1.5$). The solution is the union of $(0.5, 1)$ and $(1, 1.5)$.

69. Since $|4 - 2x| \geq 0 > -1$ for all x, the given inequality is equivalent to $|4 - 2x| < 1$, so that $-1 < 4 - 2x < 1$, or $-5 < -2x < -3$, or $\frac{5}{2} > x > \frac{3}{2}$. The solution is $(\frac{3}{2}, \frac{5}{2})$.

70. If $|x - a| \leq d$, then $-d \leq x - a \leq d$, so $a - d \leq x \leq a + d$. The solution is $[a - d, a + d]$.

71. $\dfrac{69^{800}}{59^{800}} = \left(\dfrac{69}{59}\right)^{800} \approx 2.498407507 \times 10^{54}$

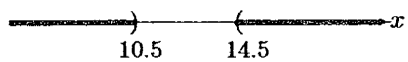
72. $\dfrac{221^{907}}{221^{897}} = 221^{907-897} = 221^{10} \approx 2.779218787 \times 10^{23}$

73. $\dfrac{(0.123)^{9000}}{(0.125)^{9000}} = \left(\dfrac{0.123}{0.125}\right)^{9000} \approx 9.034120564 \times 10^{-64}$

74. a. $$\frac{1}{\sqrt{25{,}000}-\sqrt{24{,}998}}=\frac{1}{\sqrt{25{,}000}-\sqrt{24{,}998}}\,\frac{\sqrt{25{,}000}+\sqrt{24{,}998}}{\sqrt{25{,}000}+\sqrt{24{,}998}}$$
$$=\frac{\sqrt{25{,}000}+\sqrt{24{,}998}}{25{,}000-24{,}998}=\frac{1}{2}(\sqrt{25{,}000}+\sqrt{24{,}998})$$

b. Probably the latter, since it involves no division by small numbers.

75. We desire all x such that $|x-12|+|x-13|>4$. If $x\geq 13$, then the inequality becomes $x-12+x-13>4$, that is, $2x>4+25=29$, so $x>14.5$. If $x\leq 12$, then the inequality becomes $-(x-12)-(x-13)>4$, that is, $-2x>4-25=-21$, so $x<10.5$. Finally, if $12<x<13$, then $|x-12|\leq 1$ and $|x-13|\leq 1$, so $|x-12|+|x-13|<4$. Consequently the solution is the union of $(-\infty,10.5)$ and $(14.5,\infty)$.



76. We desire all x such that $|x-2|<2|x-3|$. If $x\geq 3$, then the inequality becomes $x-2<2(x-3)=2x-6$, that is, $4<x$. If $x\leq 2$, then the inequality becomes $-(x-2)<-2(x-3)$, that is, $-x+2<-2x+6$, so that $x<4$. Finally, if $2<x<3$, then the inequality becomes $x-2<-2(x-3)=-2x+6$, that is, $3x<8$, or $x<\frac{8}{3}$. Thus the solution is the union of $(4,\infty)$, $(-\infty,2]$ and $(2,\frac{8}{3})$, which is the union of $(-\infty,\frac{8}{3})$ and $(4,\infty)$.

77. Yes, because if $x>5$, then $x^2>25$.

78. Yes, because if $x<0$, then $x^3<0$, and if $0\leq x\leq 5$, then $0\leq x^3\leq 125$.

79. No, because if $0<x\leq 1$, then $1/x\geq 1\geq x$.

80. a. If $x<0$, then we have $x<0<x^2$, so $x<x^2$. If $x>1$, then $x^2=x\cdot x>x\cdot 1=x$.

b. If $0<x<1$, then $x^2=x\cdot x<x\cdot 1=x$.

81. a. If $a\geq 0$ and $b\geq 0$ (or $a\leq 0$ and $b\leq 0$), then $|ab|=ab=|a|\,|b|$. If $a\geq 0$ and $b\leq 0$ (or $a\leq 0$ and $b\geq 0$), then $|ab|=-(ab)=|a|\,|b|$. In either case, $|ab|=|a|\,|b|$.

b. If $b\geq 0$, then $b=|b|\geq -|b|$. If $b<0$, then $-b=|b|>-|b|$, so $b=-|b|<|b|$. In either case, $-|b|\leq b\leq |b|$.

c. $|a-b|=|(-1)(b-a)|=|-1|\,|b-a|=|b-a|$.

82. a. By (9), $|b|+|c|\geq |b+c|$, so $|c|\geq |b+c|-|b|$. If $c=a-b$, then $|a-b|=|c|\geq |b+(a-b)|-|b|=|a|-|b|$.

b. Interchanging the roles of a and b in (a), we have $|b-a|\geq |b|-|a|$. From (6), $|b-a|=|a-b|$. Thus $|a-b|\geq |b|-|a|$.

c. Since $\big||a|-|b|\big|=|a|-|b|$ or $\big||a|-|b|\big|=-(|a|-|b|)=|b|-|a|$, we have from (a) and (b) that $|a-b|\geq \big||a|-|b|\big|$.

83. $(|a|+|b|)^2=|a|^2+2|a|\,|b|+|b|^2=a^2+2|ab|+b^2\geq a^2+2ab+b^2=(a+b)^2=|a+b|^2$, with equality holding if and only if $|ab|=ab$. But $|ab|=ab$ if and only if $ab\geq 0$. Thus $(|a|+|b|)^2=|a+b|^2$, and hence $|a|+|b|=|a+b|$, if and only if $ab\geq 0$.

84. If $a < b$, then $a+a < a+b < b+b$, so $a < (a+b)/2 < b$. Also $(a+b)/2$ is the midpoint of the interval $[a, b]$.

85. If $0 < a < b$, then $0 < (\sqrt{b/2} - \sqrt{a/2})^2 = (b/2 - 2\sqrt{ab/4} + a/2) = [(a+b)/2 - \sqrt{ab}]$, so $\sqrt{ab} < (a+b)/2$. Also $a = \sqrt{a^2} < \sqrt{ab}$.

86. If $1/h = \frac{1}{2}(1/a + 1/b) = \frac{1}{2}[(a+b)/ab]$, then $h = 2ab/(a+b)$. By Exercise 84, $a < (a+b)/2 < b$, so $1/b < 2/(a+b) < 1/a$, and thus $a = ab \cdot 1/b < 2ab/(a+b) = h < ab \cdot 1/a = b$.

87. If $0 < a < b$, then $a = \sqrt{aa} < \sqrt{ab}$, so $-2a > -2\sqrt{ab}$, and thus $(\sqrt{b} - \sqrt{a})^2 = b - 2\sqrt{ab} + a < b - 2a + a = b - a$. Consequently $\sqrt{b} - \sqrt{a} < \sqrt{b-a}$.

88. Assume $\sqrt{2} = p/q$, where p and q are integers such that at most one of them is divisible by 2. Then $2 = p^2/q^2$, or $p^2 = 2q^2$. Thus 2 divides p^2, so 2 divides p—say $p = 2a$. Then $2q^2 = p^2 = 4a^2$, so we have $q^2 = 2a^2$. Thus 2 divides q^2 and hence q. Therefore 2 divides p and q, contradicting our assumption. Consequently $\sqrt{2}$ is irrational.

89. Assume $\sqrt{3} = p/q$, where p and q are integers such that at most one of them is divisible by 3. Then $3 = p^2/q^2$, or $p^2 = 3q^2$. Thus 3 divides p^2, so 3 divides p—say $p = 3a$. Then $3q^2 = p^2 = 9a^2$, so we have $q^2 = 3a^2$. Thus 3 divides q^2 and hence q. Therefore 3 divides both p and q, contradicting our assumption. Consequently $\sqrt{3}$ is irrational.

90. a. Since the area equals xy, the desired inequality is $xy < 10$.
    b. Since the perimeter equals $2x + 2y$, the desired inequality is $2x + 2y \geq 47$.

91. Let a and b be adjacent sides of the rectangle. Then $P = 2(a+b)$. By Exercise 85, $\sqrt{ab} \leq (a+b)/2$, so $ab \leq [(a+b)/2]^2 = (P/4)^2$. But ab is the area of the rectangle, whereas $(P/4)^2$ is the area of a square with perimeter P.

92. The radius of the circle is $P/(2\pi)$, so the area A_C of the circle is given by $A_C = \pi[(P/2\pi)]^2 = P^2/(4\pi)$. But the area A_S of the square is given by $A_S = (P/4)^2 = P^2/16$. Since $4\pi < 16$, we have $A_S < A_C$.

93. If A_R is the area of a rectangle of perimeter P, and A_S the area of a square of perimeter P, then by Exercise 91, $A_R \leq A_S$. If A_C is the area of a circle of circumference (perimeter) P, then by Exercise 92, $A_S < A_C$. Thus $A_R < A_C$.

1.2 Points and Lines in the Plane

1.