CHAPTER 2, PART A

2.1 Given

$$\left[S_{ij}\right]=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 3 & 0 & 3\end{bmatrix} \text{ and } \left[a_i\right]=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$ (e) $a_m a_m$, (f) $S_{mn}a_m a_n$, (g) $S_{nm}a_m a_n$

Ans. (a) $S_{ii}=S_{11}+S_{22}+S_{33}=1+1+3=5$.

(b) $S_{ij}S_{ij}=S_{11}^2+S_{12}^2+S_{13}^2+S_{21}^2+S_{22}^2+S_{23}^2+S_{31}^2+S_{32}^2+S_{33}^2=$

$1+0+4+0+1+4+9+0+9=28$.

(c) $S_{ji}S_{ji}=S_{ij}S_{ij}$ =28.

(d) $S_{jk}S_{kj}=S_{1k}S_{k1}+S_{2k}S_{k2}+S_{3k}S_{k3}$

$=S_{11}S_{11}+S_{12}S_{21}+S_{13}S_{31}+S_{21}S_{12}+S_{22}S_{22}+S_{23}S_{32}+S_{31}S_{13}+S_{32}S_{23}+S_{33}S_{33}$

$=(1)(1)+(0)(0)+(2)(3)+(0)(0)+(1)(1)+(2)(0)+(3)(2)+(0)(2)+(3)(3)=23$.

(e) $a_m a_m=a_1^2+a_2^2+a_3^2=1+4+9=14$.

(f) $S_{mn}a_m a_n=S_{1n}a_1a_n+S_{2n}a_2a_n+S_{3n}a_3a_n=$

$S_{11}a_1a_1+S_{12}a_1a_2+S_{13}a_1a_3+S_{21}a_2a_1+S_{22}a_2a_2+S_{23}a_2a_3+S_{31}a_3a_1+S_{32}a_3a_2+S_{33}a_3a_3$

$=(1)(1)(1)+(0)(1)(2)+(2)(1)(3)+(0)(2)(1)+(1)(2)(2)+(2)(2)(3)+(3)(3)(1)$

$+(0)(3)(2)+(3)(3)(3)=1+0+6+0+4+12+9+0+27=59.$

(g) $S_{nm}a_m a_n=S_{mn}a_m a_n$ =59.

2.2 Determine which of these equations have an identical meaning with $a_i=Q_{ij}a_j'$.

(a) $a_p=Q_{pm}a_m'$, (b) $a_p=Q_{qp}a_q'$, (c) $a_m=a_n'Q_{mn}$.

Ans. (a) and (c)

2.3 Given the following matrices

$$\left[a_i\right]=\begin{bmatrix}1\\ 0\\ 2\end{bmatrix},\left[B_{ij}\right]=\begin{bmatrix}2 & 3 & 0\\ 0 & 5 & 1\\ 0 & 2 & 1\end{bmatrix}$$

Demonstrate the equivalence of the subscripted equations and corresponding matrix equations in the following two problems.

(a) $b_i=B_{ij}a_j$ and $[b]=[B][a]$, (b) $s=B_{ij}a_ia_j$ and $s=[a]^{\mathrm{T}}[B][a]$

Ans. (a)

$b_i=B_{ij}a_j\rightarrow b_1=B_{1j}a_j=B_{11}a_1+B_{12}a_2+B_{13}a_3=(2)(1)+(3)(0)+(0)(2)=2$

$b_2=B_{2j}a_j=B_{21}a_1+B_{22}a_2+B_{23}a_3=2,\quad b_3=B_{3j}a_j=B_{31}a_1+B_{32}a_2+B_{33}a_3=2$.

$$[b]=[B][a]=\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}.$$ Thus, $b_i=B_{ij}a_j$ gives the same results as $[b]=[B][a]$

(b)

$$s=B_{ij}a_ia_j=B_{11}a_1a_1+B_{12}a_1a_2+B_{13}a_1a_3++B_{21}a_2a_1+B_{22}a_2a_2+B_{23}a_2a_3$$
$$+B_{31}a_3a_1+B_{32}a_3a_2+B_{33}a_3a_3=(2)(1)(1)+(3)(1)(0)+(0)(1)(2)+(0)(0)(1)$$
$$+(5)(0)(0)+(1)(0)(2)+(0)(2)(1)+(2)(2)(0)+(1)(2)(2)=2+4=6.$$

and $s=[a]^{\mathrm{T}}[B][a]=\begin{bmatrix}1&0&2\end{bmatrix}\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}1&0&2\end{bmatrix}\begin{bmatrix}2\\2\\2\end{bmatrix}=2+4=6$.

2.4 Write in indicial notation the matrix equation (a) $[A]=[B][C]$, (b) $[D]=[B]^{\mathrm{T}}[C]$ and (c) $[E]=[B]^{\mathrm{T}}[C][F]$.

Ans. (a) $[A]=[B][C]\rightarrow A_{ij}=B_{im}C_{mj}$, (b) $[D]=[B]^{\mathrm{T}}[C]\rightarrow A_{ij}=B_{mi}C_{mj}$.

(c) $[E]=[B]^{\mathrm{T}}[C][F]\rightarrow E_{ij}=B_{mi}C_{mk}F_{kj}$.

2.5 Write in indicial notation the equation (a) $s=A_1^2+A_2^2+A_3^2$ and (b) $\dfrac{\partial^2\phi}{\partial x_1^2}+\dfrac{\partial^2\phi}{\partial x_2^2}+\dfrac{\partial^2\phi}{\partial x_3^2}=0$.

Ans. (a) $s=A_1^2+A_2^2+A_3^2=A_iA_i$. (b) $\dfrac{\partial^2\phi}{\partial x_1^2}+\dfrac{\partial^2\phi}{\partial x_2^2}+\dfrac{\partial^2\phi}{\partial x_3^2}=0\rightarrow\dfrac{\partial^2\phi}{\partial x_i\partial x_i}=0$.

2.6 Given that $S_{ij}=a_ia_j$ and $S'_{ij}=a'_ia'_j$, where $a'_i=Q_{mi}a_m$ and $a'_j=Q_{nj}a_n$, and $Q_{ik}Q_{jk}=\delta_{ij}$. Show that $S'_{ii}=S_{ii}$.

Ans. $S'_{ij}=Q_{mi}a_mQ_{nj}a_n=Q_{mi}Q_{nj}a_ma_n\rightarrow S'_{ii}=Q_{mi}Q_{ni}a_ma_n=\delta_{mn}a_ma_n=a_ma_m=S_{mm}=S_{ii}$.

2.7 Write $a_i=\dfrac{\partial v_i}{\partial t}+v_j\dfrac{\partial v_i}{\partial x_j}$ in long form.

Ans.

$$i=1\rightarrow a_1=\frac{\partial v_1}{\partial t}+v_j\frac{\partial v_1}{\partial x_j}=\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}.$$

$$i=2\rightarrow a_2=\frac{\partial v_2}{\partial t}+v_j\frac{\partial v_2}{\partial x_j}=\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}.$$

$$i=3\rightarrow a_3=\frac{\partial v_3}{\partial t}+v_j\frac{\partial v_3}{\partial x_j}=\frac{\partial v_3}{\partial t}+v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}.$$

2.8 Given that $T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}$, show that

(a) $T_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda\left(E_{kk}\right)^2$ and (b) $T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2)$

Ans. (a)

$$T_{ij}E_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}\delta_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}E_{ii} = 2\mu E_{ij}E_{ij} + \lambda(E_{kk})^2$$

(b)

$$T_{ij}T_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})(2\mu E_{ij} + \lambda E_{kk}\delta_{ij}) = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ij}E_{kk}\delta_{ij} + 2\mu\lambda E_{kk}\delta_{ij}E_{ij}$$

$$+\lambda^2\left(E_{kk}\right)^2\delta_{ij}\delta_{ij} = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ii}E_{kk} + 2\mu\lambda E_{kk}E_{ii} + \lambda^2\left(E_{kk}\right)^2\delta_{ii}$$

$$= 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2).$$

2.9 Given that $a_i = T_{ij}b_j$, and $a_i' = T_{ij}'b_j'$, where $a_i = Q_{im}a_m'$ and $T_{ij} = Q_{im}Q_{jn}T_{mn}'$.

(a) Show that $Q_{im}T_{mn}'b_n' = Q_{im}Q_{jn}T_{mn}'b_j$ and (b) if $Q_{ik}Q_{im} = \delta_{km}$, then $T_{kn}'(b_n' - Q_{jn}b_j) = 0$.

Ans. (a) Since $a_i = Q_{im}a_m'$ and $T_{ij} = Q_{im}Q_{jn}T_{mn}'$, therefore, $a_i = T_{ij}b_j \rightarrow$.

$Q_{im}a_m' = Q_{im}Q_{jn}T_{mn}'b_j$ (1), Now, $a_i' = T_{ij}'b_j' \rightarrow a_m' = T_{mj}'b_j' = T_{mn}'b_n'$, therefore, Eq. (1) becomes

$Q_{im}T_{mn}'b_n' = Q_{im}Q_{jn}T_{mn}'b_j$. (2)

(b) To remove Q_{im} from Eq. (2), we make use of $Q_{ik}Q_{im} = \delta_{km}$ by multiplying the above equation, Eq.(2) with Q_{ik}. That is,

$$Q_{ik}Q_{im}T_{mn}'b_n' = Q_{ik}Q_{im}Q_{jn}T_{mn}'b_j \rightarrow \delta_{km}T_{mn}'b_n' = \delta_{km}Q_{jn}T_{mn}'b_j \rightarrow T_{kn}'b_n' = Q_{jn}T_{kn}'b_j$$

$$\rightarrow T_{kn}'(b_n' - Q_{jn}b_j) = 0.$$

2.10 Given $\left[a_i\right] = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\left[b_i\right] = \begin{bmatrix}0\\2\\3\end{bmatrix}$ Evaluate $[d_i]$, if $d_k = \varepsilon_{ijk}a_ib_j$ and show that this result is the same as $d_k = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_k$.

Ans. $d_k = \varepsilon_{ijk}a_ib_j \rightarrow$

$$d_1 = \varepsilon_{ij1}a_ib_j = \varepsilon_{231}a_2b_3 + \varepsilon_{321}a_3b_2 = a_2b_3 - a_3b_2 = (2)(3) - (0)(2) = 6$$

$$d_2 = \varepsilon_{ij2}a_ib_j = \varepsilon_{312}a_3b_1 + \varepsilon_{132}a_1b_3 = a_3b_1 - a_1b_3 = (0)(0) - (1)(3) = -3$$

$$d_3 = \varepsilon_{ij3}a_ib_j = \varepsilon_{123}a_1b_2 + \varepsilon_{213}a_2b_1 = a_1b_2 - a_2b_1 = (1)(2) - (2)(0) = 2$$

Next, $\left(\mathbf{a}\times\mathbf{b}\right) = \left(\mathbf{e}_1 + 2\mathbf{e}_2\right)\times\left(2\mathbf{e}_2 + 3\mathbf{e}_3\right) = 6\mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

$d_1 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_1 = 6,\quad d_2 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_2 = -3,\quad d_3 = \left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{e}_3 = 2$.

2.11 (a) If $\varepsilon_{ijk}T_{ij} = 0$, show that $T_{ij} = T_{ji}$, and (b) show that $\delta_{ij}\varepsilon_{ijk} = 0$

Ans. (a) *for* $k=1$, $\varepsilon_{ij1}T_{ij}=0 \to \varepsilon_{231}T_{23}+\varepsilon_{321}T_{32}=0 \to T_{23}-T_{32} \to T_{23}=T_{32}$.
for $k=2$, $\varepsilon_{ij2}T_{ij}=0 \to \varepsilon_{312}T_{31}+\varepsilon_{132}T_{13}=0 \to T_{31}-T_{13} \to T_{31}=T_{13}$.
for $k=3$, $\varepsilon_{ij3}T_{ij}=0 \to \varepsilon_{123}T_{12}+\varepsilon_{213}T_{21}=0 \to T_{12}-T_{21} \to T_{12}=T_{21}$.
(b) $\delta_{ij}\varepsilon_{ijk}=\delta_{11}\varepsilon_{11k}+\delta_{22}\varepsilon_{22k}+\delta_{33}\varepsilon_{33k}=(1)(0)+(1)(0)+(1)(0)=0$.

2.12 Verify the following equation: $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$.
(Hint): there are 6 cases to be considered (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$.

Ans. There are 4 free indices in the equation. Therefore, there are the following 6 cases to consider: (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$. We consider each case below where we use LS for left side, RS for right side and repeated indices with parenthesis are not sum:
(1) For $i=j$, LS=$\varepsilon_{(i)(i)m}\varepsilon_{klm}=0$, $RS=\delta_{(i)k}\delta_{(i)l}-\delta_{(i)l}\delta_{(i)k}=0$.
(2) For $i=k$, LS=$\varepsilon_{(i)j1}\varepsilon_{(i)l1}+\varepsilon_{(i)j2}\varepsilon_{(i)l2}+\varepsilon_{(i)j3}\varepsilon_{(i)l3}$, $RS=\delta_{(i)(i)}\delta_{jl}-\delta_{(i)l}\delta_{j(i)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq l\\ 0 & \text{if } j=l=i\\ 1 & \text{if } j=l\neq i\end{cases}.$$

(3) For $i=l$, LS=$\varepsilon_{(i)jm}\varepsilon_{k(i)m}$, $RS=\delta_{(i)k}\delta_{j(i)}-\delta_{(i)(i)}\delta_{jk}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq k\\ 0 & \text{if } j=k=i\\ -1 & \text{if } j=k\neq i\end{cases}$$

(4) For $j=k$, LS=$\varepsilon_{i(j)m}\varepsilon_{(j)lm}$, $RS=\delta_{i(j)}\delta_{(j)l}-\delta_{il}\delta_{(j)(j)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq l\\ 0 & \text{if } i=l=j\\ -1 & \text{if } i=l\neq j\end{cases}$$

(5) For $j=l$, LS=$\varepsilon_{i(j)m}\varepsilon_{k(j)m}$, $RS=\delta_{ik}\delta_{(j)(j)}-\delta_{i(j)}\delta_{(j)k}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq k\\ 0 & \text{if } i=k=j\\ 1 & \text{if } i=k\neq j\end{cases}$$

(6) For $k=l$, LS=$\varepsilon_{ijm}\varepsilon_{(k)(k)m}$=0, $RS=\delta_{i(k)}\delta_{j(k)}-\delta_{i(k)}\delta_{j(k)}=0$

2.13 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ as a short cut to obtain the following results: (a) $\varepsilon_{ilm}\varepsilon_{jlm}=2\delta_{ij}$ and (b) $\varepsilon_{ijk}\varepsilon_{ijk}=6$.

Ans. (a) $\varepsilon_{ilm}\varepsilon_{jlm}=\delta_{ij}\delta_{ll}-\delta_{il}\delta_{lj}=3\delta_{ij}-\delta_{ij}=2\delta_{ij}$.
(b) $\varepsilon_{ijk}\varepsilon_{ijk}=\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ji}=(3)(3)-\delta_{ii}=9-3=6$.

2.14 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ to show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Ans. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = a_m\mathbf{e}_m \times (\varepsilon_{ijk}b_jc_k\mathbf{e}_i) = \varepsilon_{ijk}a_mb_jc_k(\mathbf{e}_m\times\mathbf{e}_i)$

$= \varepsilon_{ijk}a_mb_jc_k(\varepsilon_{nmi}\mathbf{e}_n) = \varepsilon_{ijk}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n = \varepsilon_{jki}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n$

$= (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})a_mb_jc_k\mathbf{e}_n = \delta_{jn}\delta_{km}a_mb_jc_k\mathbf{e}_n - \delta_{jm}\delta_{kn}a_mb_jc_k\mathbf{e}_n$

$= a_kb_nc_k\mathbf{e}_n - a_jb_jc_n\mathbf{e}_n = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

2.15 (a) Show that if $T_{ij} = -T_{ji}$, $T_{ij}a_ia_j = 0$ and (b) if $T_{ij} = -T_{ji}$, and $S_{ij} = S_{ji}$, then $T_{ij}S_{ij} = 0$

Ans. Since $T_{ij}a_ia_j = T_{ji}a_ja_i$ (switching the original dummy index i to j and the original index j to i), therefore $T_{ij}a_ia_j = T_{ji}a_ja_i = -T_{ij}a_ja_i = -T_{ij}a_ia_j \to 2T_{ij}a_ia_j = 0 \to T_{ij}a_ia_j = 0$.

(b) $T_{ij}S_{ij} = T_{ji}S_{ji}$ (switching the original dummy index i to j and the original index j to i), therefore, $T_{ij}S_{ij} = T_{ji}S_{ji} = -T_{ij}S_{ji} = -T_{ij}S_{ij} \to 2T_{ij}S_{ij} = 0 \to T_{ij}S_{ij} = 0$.

2.16 Let $T_{ij} = \left(S_{ij} + S_{ji}\right)/2$ and $R_{ij} = \left(S_{ij} - S_{ji}\right)/2$, show that $T_{ij} = T_{ji}$, $R_{ij} = -R_{ji}$, and $S_{ij} = T_{ij} + \mathrm{R}_{ij}$.

Ans. $T_{ij} = \left(S_{ij} + S_{ji}\right)/2 \to T_{ji} = \left(S_{ji} + S_{ij}\right)/2 = T_{ij}$.

$R_{ij} = \left(S_{ij} - S_{ji}\right)/2 \to R_{ji} = \left(S_{ji} - S_{ij}\right)/2 = -\left(S_{ij} - S_{ji}\right)/2 = -R_{ij}$.

$T_{ij} + R_{ij} = \left(S_{ij} + S_{ji}\right)/2 + \left(S_{ij} - S_{ji}\right)/2 = S_{ij}$.

2.17 Let $f(x_1, x_2, x_3)$ be a function of $x_1, x_2, \text{and } x_3$ and $v_i(x_1, x_2, x_3)$ be three functions of $x_1, x_2, \text{and } x_3$. Express the total differential df and dv_i in indicial notation.

Ans. $df = \dfrac{\partial f}{\partial x_1}dx_1 + \dfrac{\partial f}{\partial x_2}dx_2 + \dfrac{\partial f}{\partial x_3}dx_3 = \dfrac{\partial f}{\partial x_i}dx_i$.

$dv_i = \dfrac{\partial v_i}{\partial x_1}dx_1 + \dfrac{\partial v_i}{\partial x_2}dx_2 + \dfrac{\partial v_i}{\partial x_3}dx_3 = \dfrac{\partial v_i}{\partial x_m}dx_m$.

2.18 Let $\left|A_{ij}\right|$ denote that determinant of the matrix $\left[A_{ij}\right]$. Show that $\left|A_{ij}\right| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

Ans. $\varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \varepsilon_{1jk}A_{11}A_{j2}A_{k3} + \varepsilon_{2jk}A_{21}A_{j2}A_{k3} + \varepsilon_{3jk}A_{31}A_{j2}A_{k3}$

$= \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{132}A_{11}A_{32}A_{23} + \varepsilon_{231}A_{21}A_{32}A_{13} + \varepsilon_{213}A_{21}A_{12}A_{33} + \varepsilon_{312}A_{31}A_{12}A_{23} + \varepsilon_{321}A_{31}A_{22}A_{13}$

$= A_{11}A_{22}A_{33} - A_{11}A_{32}A_{23} + A_{21}A_{32}A_{13} - A_{21}A_{12}A_{33} + A_{31}A_{12}A_{23} - A_{31}A_{22}A_{13}$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$
