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# CLASSICAL MECHANICS SOLUTIONS MANUAL

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Please report any errors in these solutions  
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## **Chapter One**

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### **The algebra and calculus of vectors**

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**Problem 1.1**

In terms of the standard basis set  $\{i, j, k\}$ ,  $a = 2i - j - 2k$ ,  $b = 3i - 4k$  and  $c = i - 5j + 3k$ .

- (i) Find  $3a + 2b - 4c$  and  $|a - b|^2$ .
- (ii) Find  $|a|$ ,  $|b|$  and  $a \cdot b$ . Deduce the angle between  $a$  and  $b$ .
- (iii) Find the component of  $c$  in the direction of  $a$  and in the direction of  $b$ .
- (iv) Find  $a \times b$ ,  $b \times c$  and  $(a \times b) \times (b \times c)$ .
- (v) Find  $a \cdot (b \times c)$  and  $(a \times b) \cdot c$  and verify that they are equal. Is the set  $\{a, b, c\}$  right- or left-handed?
- (vi) By evaluating each side, verify the identity  $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ .

**Solution**

(i)

$$\begin{aligned} 3a + 2b - 4c &= 3(2i - j - 2k) + 2(3i - 4k) - 4(i - 5j + 3k) \\ &= 8i + 17j - 26k. \blacksquare \end{aligned}$$

$$\begin{aligned} |a - b|^2 &= (a - b) \cdot (a - b) \\ &= (-i - j + 2k) \cdot (-i - j + 2k) \\ &= (-1)^2 + (-1)^2 + 2^2 = 6. \blacksquare \end{aligned}$$

(ii)

$$\begin{aligned} |a|^2 &= a \cdot a \\ &= (2i - j - 2k) \cdot (2i - j - 2k) \\ &= 2^2 + (-1)^2 + (-2)^2 = 9. \end{aligned}$$

Hence  $|a| = 3$ .  $\blacksquare$ 

$$\begin{aligned} |b|^2 &= b \cdot b \\ &= (3i - 4k) \cdot (3i - 4k) \\ &= 3^2 + (-4)^2 = 25. \end{aligned}$$

Hence  $|b| = 5$ .  $\blacksquare$ 

$$\begin{aligned} a \cdot b &= (2i - j - 2k) \cdot (3i - 4k) \\ &= (2 \times 3) + ((-1) \times 0) + ((-2) \times (-4)) \\ &= 14. \blacksquare \end{aligned}$$

The angle  $\alpha$  between  $\mathbf{a}$  and  $\mathbf{b}$  is then given by

$$\begin{aligned}\cos \alpha &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ &= \frac{14}{3 \times 5} = \frac{14}{15}.\end{aligned}$$

Thus  $\alpha = \tan^{-1} \frac{14}{15}$ . ■

(iii) The component of  $\mathbf{c}$  in the direction of  $\mathbf{a}$  is

$$\begin{aligned}\mathbf{c} \cdot \hat{\mathbf{a}} &= \mathbf{c} \cdot \left( \frac{\mathbf{a}}{|\mathbf{a}|} \right) \\ &= (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \cdot \left( \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}|} \right) \\ &= \frac{(1 \times 2) + ((-5) \times (-1)) + (3 \times (-2))}{3} \\ &= \frac{1}{3}. \quad \blacksquare\end{aligned}$$

The component of  $\mathbf{c}$  in the direction of  $\mathbf{b}$  is

$$\begin{aligned}\mathbf{c} \cdot \hat{\mathbf{b}} &= \mathbf{c} \cdot \left( \frac{\mathbf{b}}{|\mathbf{b}|} \right) \\ &= (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \cdot \left( \frac{3\mathbf{i} - 4\mathbf{k}}{|3\mathbf{i} - 4\mathbf{k}|} \right) \\ &= \frac{(1 \times 3) + ((-5) \times 0) + (3 \times (-4))}{5} \\ &= -\frac{9}{5}. \quad \blacksquare\end{aligned}$$

(iv)

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times (3\mathbf{i} - 4\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 3 & 0 & -4 \end{vmatrix} \\ &= (4 - 0)\mathbf{i} - ((-8) - (-6))\mathbf{j} + (0 - (-3))\mathbf{k} \\ &= 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}. \quad \blacksquare\end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \times \mathbf{c} &= (3\mathbf{i} - 4\mathbf{k}) \times (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -4 \\ 1 & -5 & 3 \end{vmatrix} \\
 &= (0 - 20)\mathbf{i} - (9 - (-4))\mathbf{j} + ((-15) - 0)\mathbf{k} \\
 &= -20\mathbf{i} - 13\mathbf{j} - 15\mathbf{k}. \blacksquare
 \end{aligned}$$

Hence

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c}) &= (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (-20\mathbf{i} - 13\mathbf{j} - 15\mathbf{k}) \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 3 \\ -20 & -13 & -15 \end{vmatrix} \\
 &= ((-30) - (-39))\mathbf{i} - ((-60) - (-60))\mathbf{j} + ((-52) - (-40))\mathbf{k} \\
 &= 9\mathbf{i} - 12\mathbf{k}. \blacksquare
 \end{aligned}$$

(v)

$$\begin{aligned}
 \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \cdot (-20\mathbf{i} - 13\mathbf{j} - 15\mathbf{k}) \\
 &= (2 \times (-20)) + ((-1) \times (-13)) + ((-2) \times (-15)) \\
 &= 3.
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \\
 &= (4 \times 1) + (2 \times (-5)) + (3 \times 3) \\
 &= 3.
 \end{aligned}$$

These values are equal and this **verifies the identity**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Since  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  is *positive*, the set  $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$  must be **right-handed**.  $\blacksquare$

(vi) The **left side** of the identity is

$$\begin{aligned}
 \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times (-20\mathbf{i} - 13\mathbf{j} - 15\mathbf{k}) \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ -20 & -13 & -15 \end{vmatrix} \\
 &= (15 - 26)\mathbf{i} - ((-30) - 40)\mathbf{j} + ((-26) - 20)\mathbf{k} \\
 &= -11\mathbf{i} + 70\mathbf{j} - 46\mathbf{k}.
 \end{aligned}$$

Since

$$\begin{aligned}(a \cdot c)b &= \left( (2 \times 1) + ((-1) \times (-5)) + ((-2) \times 3) \right)b \\ &= b \\ &= 3i - 4k,\end{aligned}$$

$$\begin{aligned}(a \cdot b)c &= \left( (2 \times 3) + ((-1) \times 0) + ((-2) \times (-4)) \right)c \\ &= 14c = 14(i - 5j + 3k) \\ &= 14i - 70j + 42k,\end{aligned}$$

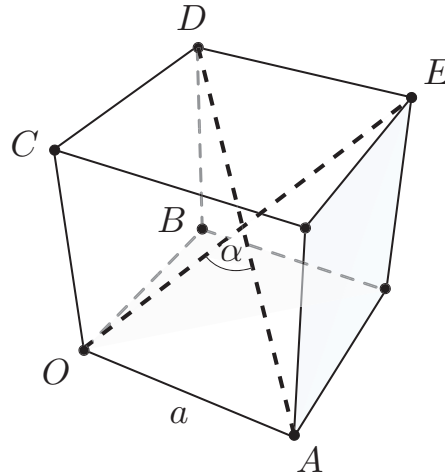
the **right side** of the identity is

$$\begin{aligned}(a \cdot c)b - (a \cdot b)c &= (3i - 4k) - (14i - 70j + 42k) \\ &= -11i + 70j - 46k.\end{aligned}$$

Thus the right and left sides are equal and this **verifies the identity**. ■

**Problem 1.2**

Find the angle between any two diagonals of a cube.



**FIGURE 1.1** Two diagonals of a cube.

**Solution**

Figure 1.1 shows a cube of side  $a$ ;  $OE$  and  $AD$  are two of its diagonals. Let  $O$  be the origin of position vectors and suppose the points  $A$ ,  $B$  and  $C$  have position vectors  $a\mathbf{i}$ ,  $a\mathbf{j}$ ,  $a\mathbf{k}$  respectively. Then the line segment  $\overrightarrow{OE}$  represents the vector

$$a\mathbf{i} + a\mathbf{j} + a\mathbf{k}$$

and the line segment  $\overrightarrow{AD}$  represents the vector

$$(a\mathbf{j} + a\mathbf{k}) - a\mathbf{i} = -a\mathbf{i} + a\mathbf{j} + a\mathbf{k}.$$

Let  $\alpha$  be the angle between  $OE$  and  $AD$ . Then

$$\begin{aligned} \cos \alpha &= \frac{(a\mathbf{i} + a\mathbf{j} + a\mathbf{k}) \cdot (-a\mathbf{i} + a\mathbf{j} + a\mathbf{k})}{|a\mathbf{i} + a\mathbf{j} + a\mathbf{k}| | -a\mathbf{i} + a\mathbf{j} + a\mathbf{k} |} \\ &= \frac{-a^2 + a^2 + a^2}{(\sqrt{3}a)(\sqrt{3}a)} = \frac{1}{3}. \end{aligned}$$

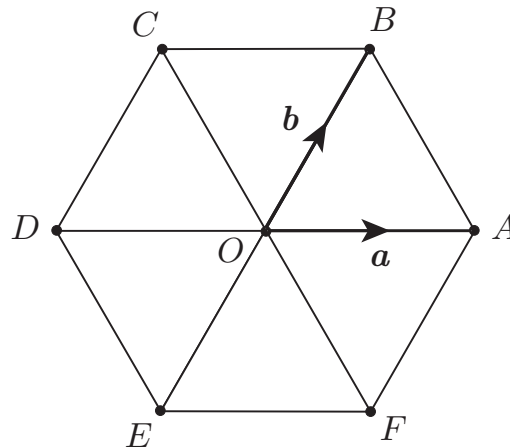
Hence the **angle between the diagonals** is  $\cos^{-1} \frac{1}{3}$ , which is approximately  $70.5^\circ$ .

■



**Problem 1.3**

$ABCDEF$  is a regular hexagon with centre  $O$  which is also the origin of position vectors. Find the position vectors of the vertices  $C, D, E, F$  in terms of the position vectors  $\mathbf{a}, \mathbf{b}$  of  $A$  and  $B$ .



**FIGURE 1.2**  $ABCDEF$  is a regular hexagon.

**Solution**

- (i) The position vector  $\mathbf{c}$  is represented by the line segment  $\overrightarrow{OC}$  which has the same magnitude and direction as the line segment  $\overrightarrow{AB}$ . Hence

$$\mathbf{c} = \mathbf{b} - \mathbf{a}. \blacksquare$$

- (ii) The position vector  $\mathbf{d}$  is represented by the line segment  $\overrightarrow{OD}$  which has the same magnitude as, but *opposite* direction to, the line segment  $\overrightarrow{OA}$ . Hence

$$\mathbf{d} = -\mathbf{a}. \blacksquare$$

- (iii) The position vector  $\mathbf{e}$  is represented by the line segment  $\overrightarrow{OE}$  which has the same magnitude as, but *opposite* direction to, the line segment  $\overrightarrow{OB}$ . Hence

$$\mathbf{e} = -\mathbf{b}. \blacksquare$$

- (iv) The position vector  $\mathbf{f}$  is represented by the line segment  $\overrightarrow{OF}$  which has the

same magnitude as, but *opposite* direction to, the line segment  $\overrightarrow{AB}$ . Hence

$$\mathbf{e} = -(\mathbf{b} - \mathbf{a}) = \mathbf{a} - \mathbf{b}. \blacksquare$$