CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1–12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$ so

$$y_1'' = -4\cos 2x = -4y_1 \quad \text{and} \quad y_2'' = -4\sin 2x = -4y_2.$$

Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$ so

$$y_1'' = 9e^{3x} = 9y_1 \quad \text{and} \quad y_2'' = 9e^{-3x} = 9y_2.$$

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$ so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

10. Integration of $y' = x e^{-x}$ yields

$$y(x) = \int x e^{-x}\,dx = \int u e^{u}\,du = (u-1)e^{u} = -(x+1)e^{-x} + C$$

(when we substitute $u = -x$ and apply Formula #46 inside the back cover of the textbook). Then substitution of $x = 0$, $y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2.$

11. If $a(t) = 50$ then $v(t) = \int 50\,dt = 50t + v_0 = 50t + 10.$ Hence

$$x(t) = \int (50t+10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$ then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15.$ Hence

$$x(t) = \int (-20t-15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$ then $v(t) = \int 3t\,dt = \tfrac{3}{2}t^2 + v_0 = \tfrac{3}{2}t^2 + 5.$ Hence

$$x(t) = \int (\tfrac{3}{2}t^2 + 5)\,dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t + 1$ then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7.$ Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 4(t+3)^2$. then $v(t) = \int 4(t+3)^2\,dt = \tfrac{4}{3}(t+3)^3 + C = \tfrac{4}{3}(t+3)^3 - 37$ (taking $C = -37$ so that $v(0) = -1$). Hence

$$x(t) = \int \left[\tfrac{4}{3}(t+3)^3 - 37\right]dt = \tfrac{1}{3}(t+3)^4 - 37t + C = \tfrac{1}{3}(t+3)^4 - 37t - 26.$$

16. If $a(t) = 1/\sqrt{t+4}$ then $v(t) = \int 1/\sqrt{t+4}\,dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int (2\sqrt{t+4} - 5)\,dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

$$\tan^{-1}(v/160) = -t/5+C; \quad v(0)=160 \text{ implies } C=\pi/4$$

$$v(t) = 160\tan\left(\frac{\pi}{4}-\frac{t}{5}\right)$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4}-\frac{t}{5}\right)\right)+400\ln 2$$

We solve $v(t) = 0$ for $t=3.92699$ and then calculate $y(3.92699) = 277.26$ ft.

21. Equation: $v' = -g-\rho v^2, \quad v(0) = v_0, \quad y(0) = 0$

Solution: $\displaystyle\int\frac{dv}{g+\rho v^2} = -\int dt; \quad \int\frac{\sqrt{\rho/g}\,dv}{1+\left(\sqrt{\rho/g}\,v\right)^2} = -\int\sqrt{g\rho}\,dt;$

$$\tan^{-1}\left(\sqrt{\rho/g}\,v\right) = -\sqrt{g\rho}\,t+C; \quad v(0)=v_0 \text{ implies } C=\tan^{-1}\left(\sqrt{\rho/g}\,v_0\right)$$

$$v(t) = -\sqrt{\frac{g}{\rho}}\tan\left(t\sqrt{g\rho}-\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)$$

We solve $v(t) = 0$ for $t = \dfrac{1}{\sqrt{g\rho}}\tan^{-1}\left(v_0\sqrt{\dfrac{\rho}{g}}\right)$ and substitute in Eq. (17) for $y(t)$:

$$\begin{aligned}
y_{\max} &= \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}-\tan^{-1}v_0\sqrt{\rho/g}\right)}{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}\right)}\right| \\
&= \frac{1}{\rho}\ln\left(\sec\left(\tan^{-1}v_0\sqrt{\rho/g}\right)\right) = \frac{1}{\rho}\ln\sqrt{1+\frac{\rho v_0^2}{g}} \\
y_{\max} &= \frac{1}{2\rho}\ln\left(1+\frac{\rho v_0^2}{g}\right)
\end{aligned}$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem $v' = -32+0.075v^2, \quad v(0)=0$ is

$$v(t) = -20.666\tanh(1.54919t),$$

so the terminal speed is 20.666 ft/sec. Then a further integration with $y(0)=0$ gives

$$y(t) = 10000-13.333\ln(\cosh(1.54919t)).$$

We solve $y(0)=0$ for $t=484.57$. Thus the descent takes about 8 min 5 sec.

(b) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with characteristic equation $(\lambda-1)^2 = 0$ and the single eigenvalue $\lambda = 1$. Then $\mathbf{A}-\mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and it follows that the only associated eigenvector is a multiple of $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$. The transpose $\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic equation and eigenvalue, but we see similarly that its only eigenvector is a multiple of $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Thus $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalue but different eigenvectors.

36. If the $n \times n$ matrix $\mathbf{A} = \left[a_{ij} \right]$ is either upper or lower triangular, then obviously its characteristic equation is

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots\cdot(a_{nn}-\lambda) = 0.$$

This observation makes it clear that the eigenvalues of the matrix $\mathbf{A}$ are its diagonal elements $a_{11}, a_{22}, \cdots, a_{nn}$.

37. If $|\mathbf{A}-\lambda\mathbf{I}| = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$, then substitution of $\lambda = 0$ yields $c_0 = |\mathbf{A}-0\mathbf{I}| = |\mathbf{A}|$ for the constant term in the characteristic polynomial.

38. The characteristic polynomial of the 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $(a-\lambda)(d-\lambda)-bc = 0$, that is, $\lambda^2-(a+d)\lambda+(ad-bc) = 0$. Thus the coefficient of λ in the characteristic equation is $-(a+d) = -\operatorname{trace}\mathbf{A}$.

39. If the characteristic equation of the $n\times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (not necessarily distinct) is written in the factored form

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots\cdot(\lambda-\lambda_n) = 0,$$

then it should be clear that upon multiplying out the factors the coefficient of λ^{n-1} will be $-(\lambda_1+\lambda_2+\cdots+\lambda_n)$. But according to Problem 38, this coefficient also equals $-(\operatorname{trace}\mathbf{A})$. Therefore $\lambda_1+\lambda_2+\cdots+\lambda_n = \operatorname{trace}\mathbf{A} = a_{11}+a_{22}+\cdots a_{nn}$.

40. We find that $\operatorname{trace}\mathbf{A} = 12$ and $\det\mathbf{A} = 60$, so the characteristic polynomial of the given matrix $\mathbf{A}$ is

$$p(\lambda) = -\lambda^3+12\lambda^2+c_1\lambda+60.$$

At $(n\pi,0)$, n even: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}$ has characteristic equation $\lambda^2-5\lambda+5=0$ and positive real eigenvalues $\lambda_1\approx 1.3812,\ \lambda_2=2.6180$. Hence $(n\pi,0)$ is a nodal source if n is even, as we see in the figure on the preceding page.

At $(n\pi,0)$, n odd: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-3 & 1\\ -1 & 2\end{bmatrix}$ has characteristic equation $\lambda^2+\lambda-5=0$ and real eigenvalues $\lambda_1\approx -2.7913,\ \lambda_2=1.7913$ of opposite sign. Hence $(n\pi,0)$ is a saddle point if n is odd, as we see in the figure.

As preparation for Problems 9–11, we first calculate the Jacobian matrix

$$\mathbf{J}(x,y)=\begin{bmatrix}0 & 1\\ -\omega^2\cos x & -c\end{bmatrix}$$

of the damped pendulum system in (34) in the text. At the critical point $(n\pi,0)$ we have

$$\mathbf{J}(n\pi,0)\ =\ \begin{bmatrix}0 & 1\\ -\omega^2\cos n\pi & -c\end{bmatrix}=\begin{bmatrix}0 & 1\\ \pm\omega^2 & -c\end{bmatrix},$$

where we take the plus sign if n is odd, the minus sign if n is even.

9. If n is odd then the characteristic equation $\lambda^2+c\lambda-\omega^2=0$ has real roots

$$\lambda_1,\lambda_2\ =\ \frac{-c\pm\sqrt{c^2+4\omega^2}}{2}$$

with opposite signs, so $(n\pi,\,0)$ is an unstable saddle point.

10. If n is even then the characteristic equation $\lambda^2+c\lambda+\omega^2\ =\ 0$ has roots

$$\lambda_1,\lambda_2\ =\ \frac{-c\pm\sqrt{c^2-4\omega^2}}{2}.$$

If $c^2>4\omega^2$ then λ_1 and λ_2 are both negative so $(n\pi,\,0)$ is a stable nodal sink.

11. If n is even and $c^2<4\omega^2$ then the two eigenvalues

$$\lambda_1,\lambda_2\ =\ \frac{-c\pm\sqrt{c^2-4\omega^2}}{2}\ =\ -\frac{c}{2}\pm\frac{i}{2}\sqrt{4\omega^2-c^2}$$

are complex conjugates with negative real part, so $(n\pi,\,0)$ is a stable spiral point.