Student Solutions Manual

for use with

Complex Variables and Applications

Seventh Edition

Selected Solutions to Exercises in Chapters 1-7

by

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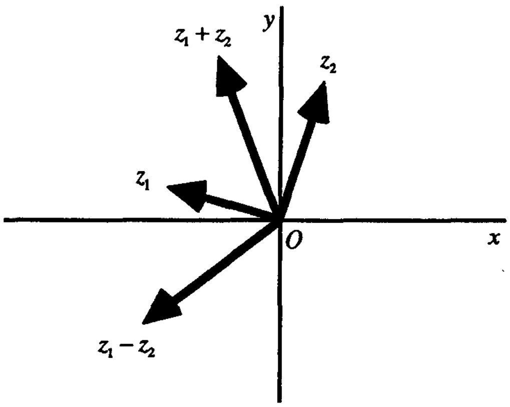
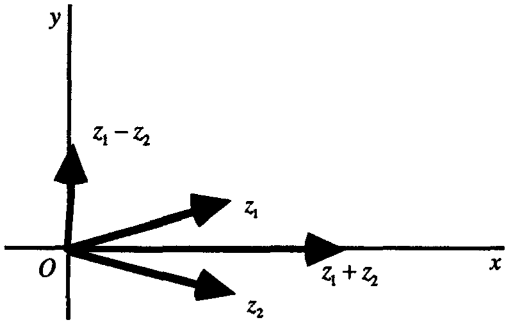
(c) $z_1 = (-3,1), \quad z_2 = (1,4)$



(d) $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$



2. Inequalities (3), Sec. 4, are

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|.$$

These are obvious if we write them as

$$x \le |x| \le \sqrt{x^2 + y^2} \quad \text{and} \quad y \le |y| \le \sqrt{x^2 + y^2}.$$

3. In order to verify the inequality $\sqrt{2}|z| \ge |\operatorname{Re} z| + |\operatorname{Im} z|$, we rewrite it in the following ways:

$$\sqrt{2}\sqrt{x^2 + y^2} \ge |x| + |y|,$$

$$2(x^2 + y^2) \ge |x|^2 + 2|x||y| + |y|^2,$$

$$|x|^2 - 2|x||y| + |y|^2 \ge 0,$$

$$(|x| - |y|)^2 \ge 0.$$

This last form of the inequality to be verified is obviously true since the left-hand side is a perfect square.

6. The four roots of the equation $z^4+4=0$ are the four fourth roots of the number -4. To find those roots, we write $-4=4\exp[i(\pi+2k\pi)]$ $(k=0,\pm1,\pm2,\ldots)$. Then

$$(-4)^{1/4}=\sqrt{2}\exp\left[i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\right]=\sqrt{2}e^{i\pi/4}e^{ik\pi/2} \qquad (k=0,1,2,3).$$

To be specific,

$$c_0=\sqrt{2}e^{i\pi/4}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=1+i,$$

$$c_1=c_0e^{i\pi/2}=(1+i)i=-1+i,$$

$$c_2=c_0e^{i\pi}=(1+i)(-1)=-1-i,$$

$$c_3=c_0e^{i3\pi/2}=(1+i)(-i)=1-i.$$

This enables us to write

$$\begin{aligned}
z^4+4&=(z-c_0)(z-c_1)(z-c_2)(z-c_3)\\
&=[(z-c_1)(z-c_2)]\cdot[(z-c_0)(z-c_3)]\\
&=[(z+1)-i][(z+1)+i]\cdot[(z-1)-i][(z-1)+i]\\
&=\left[(z+1)^2+1\right]\cdot\left[(z-1)^2+1\right]\\
&=\left(z^2+2z+2\right)\left(z^2-2z+2\right).
\end{aligned}$$

7. Let c be any nth root of unity other than unity itself. With the aid of the identity (see Exercise 10, Sec. 7),

$$1+z+z^2+\cdots+z^{n-1}=\frac{1-z^n}{1-z} \qquad (z\neq1),$$

we find that

$$1+c+c^2+\cdots+c^{n-1}=\frac{1-c^n}{1-c}=\frac{1-1}{1-c}=0.$$

9. Observe first that

$$(z^{1/m})^{-1}=\left[\sqrt[m]{r}\exp\frac{i(\theta+2k\pi)}{m}\right]^{-1}=\frac{1}{\sqrt[m]{r}}\exp\frac{i(-\theta-2k\pi)}{m}=\frac{1}{\sqrt[m]{r}}\exp\frac{i(-\theta)}{m}\exp\frac{i(-2k\pi)}{m}$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi\ (n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \qquad (n = 0 \pm 1, \pm 2, \ldots).$$

So there is no neighborhood throughout which f is analytic, and this means that $\cos\bar{z}$ is nowhere analytic.

16. *(a)* Use expression (12), Sec. 33, to write

$$\overline{\cos(iz)} = \overline{\cos(-y + ix)} = \cos y \cosh x - i \sin y \sinh x$$

and

$$\cos(i\bar{z}) = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x.$$

This shows that $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z.

(b) Use expression (11), Sec. 33, to write

$$\overline{\sin(iz)} = \overline{\sin(-y + ix)} = -\sin y \cosh x - i \cos y \sinh x$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Evidently, then, the equation $\overline{\sin(iz)} = \sin(i\bar{z})$ is equivalent to the pair of equations

$$\sin y \cosh x = 0, \quad \cos y \sinh x = 0.$$

Since $\cosh x$ is never zero, the first of these equations tells us that $\sin y = 0$. Consequently, $y = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi = (-1)^n \neq 0$, the second equation tells us that $\sinh x = 0$, or that $x = 0$. So we may conclude that $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = 0 + in\pi = n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

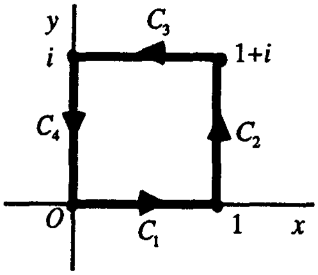
17. Rewriting the equation $\sin z = \cosh 4$ as $\sin x \cosh y + i \cos x \sinh y = \cosh 4$, we see that we need to solve the pair of equations

$$\sin x \cosh y = \cosh 4, \quad \cos x \sinh y = 0$$

(b) Here $C: z = x\ (0 \le x \le 2)$. Then

$$\int_C (z-1)\,dz = \int_0^2 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_0^2 = 0.$$

3. In this problem, the path C is the sum of the paths C_1, C_2, C_3, and C_4 that are shown below.



The function to be integrated around the closed path C is $f(z) = \pi e^{\pi \bar{z}}$. We observe that $C = C_1 + C_2 + C_3 + C_4$ and find the values of the integrals along the individual legs of the square C.

(i) Since C_1 is $z = x\ (0 \le x \le 1)$,

$$\int_{C_1} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi x} dx = e^{\pi} - 1.$$

(ii) Since C_2 is $z = 1 + iy\ (0 \le y \le 1)$,

$$\int_{C_2} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi(1-iy)} i\,dy = e^{\pi} \pi i \int_0^1 e^{-i\pi y} dy = 2e^{\pi}.$$

(iii) Since C_3 is $z = (1-x) + i\ (0 \le x \le 1)$,

$$\int_{C_3} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{\pi[(1-x)-i]}(-1)\,dx = \pi e^{\pi} \int_0^1 e^{-\pi x} dx = e^{\pi} - 1.$$

(iv) Since C_4 is $z = i(1-y)\ (0 \le y \le 1)$,

$$\int_{C_4} \pi e^{\pi \bar{z}} dz = \pi \int_0^1 e^{-\pi(1-y)i}(-i)\,dy = \pi i \int_0^1 e^{i\pi y} dy = -2.$$

Finally, then, since

$$\int_C \pi e^{\pi \bar{z}} dz = \int_{C_1} \pi e^{\pi \bar{z}} dz + \int_{C_2} \pi e^{\pi \bar{z}} dz + \int_{C_3} \pi e^{\pi \bar{z}} dz + \int_{C_4} \pi e^{\pi \bar{z}} dz,$$

we find that

$$\int_C \pi e^{\pi \bar{z}} dz = 4(e^{\pi} - 1).$$

2. *(b)* Replacing z by $z-1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty),$$

we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

So

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

3. We want to find the Maclaurin series for the function

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{1 + (z^4/9)}.$$

To do this, we first replace z by $-(z^4/9)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

as well as its condition of validity, to get

$$\frac{1}{1 + (z^4/9)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} \qquad (|z| < \sqrt{3}).$$

Then, if we multiply through this last equation by $\frac{z}{9}$, we have the desired expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \qquad (|z| < \sqrt{3}).$$

6. Replacing z by z^2 in the representation

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty),$$

we have

$$\sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \qquad (|z| < \infty).$$

(b) When $f(z) = \dfrac{1}{1+z^2}$, we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - \cdots \qquad (0 < |z| < 1).$$

Thus

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(0) = 0.$$

(c) If $f(z) = \dfrac{1}{z}$, it follows that $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z}$. Evidently, then,

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i(1) = 2\pi i.$$

4. Let C denote the circle $|z| = 1$, taken counterclockwise.

(a) The Maclaurin series $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(|z| < \infty)$ enables us to write

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = \int_C e^z e^{1/z}\,dz = \int_C e^{1/z} \sum_{n=0}^{\infty} \frac{z^n}{n!}\,dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Referring to the Maclaurin series for e^z once again, let us write

$$z^n \exp\left(\frac{1}{z}\right) = z^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

Now the $\dfrac{1}{z}$ in this series occurs when $n - k = -1$, or $k = n + 1$. So, by the residue theorem,

$$\int_C z^n \exp\left(\frac{1}{z}\right) dz = 2\pi i \frac{1}{(n+1)!} \qquad (n = 0, 1, 2, \ldots).$$

The final result in part *(a)* thus reduces to

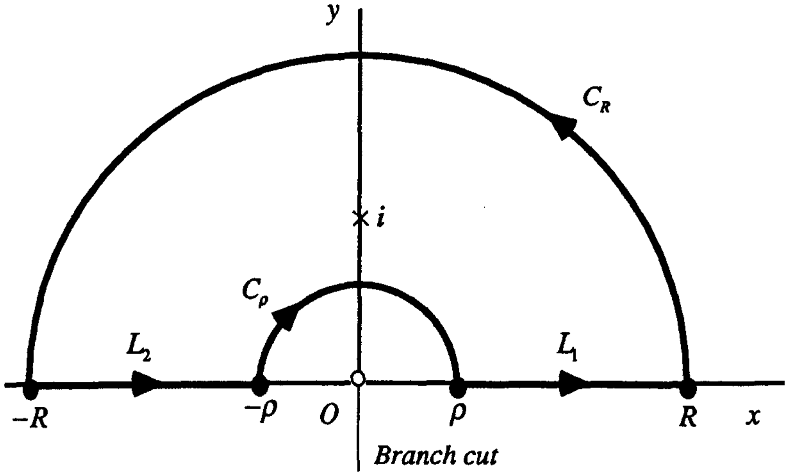
$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

6. *(a)* Let us first use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_\rho} f(z)\,dz=2\pi i\operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

Since

$$L_1: z=re^{i0}=r\ (\rho\le r\le R) \quad\text{and}\quad -L_2: z=re^{i\pi}=-r\ (\rho\le r\le R),$$

we may write

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}-i\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}=(1-i)\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$