DIAGNOSTIC TESTS

Test A Algebra

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

   (b) $-3^4 = -(3)(3)(3)(3) = -81$

   (c) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

   (d) $\dfrac{5^{23}}{5^{21}} = 5^{23-21} = 5^2 = 25$

   (e) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

   (f) $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

2. (a) Note that $\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$ and $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. Thus $\sqrt{200} - \sqrt{32} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}$.

   (b) $(3a^3b^3)(4ab^2)^2 = 3a^3b^3 16a^2b^4 = 48a^5b^7$

   (c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \left(\dfrac{x^2y^{-1/2}}{3x^{3/2}y^3}\right)^2 = \dfrac{(x^2y^{-1/2})^2}{(3x^{3/2}y^3)^2} = \dfrac{x^4y^{-1}}{9x^3y^6} = \dfrac{x^4}{9x^3y^6y} = \dfrac{x}{9y^7}$

3. (a) $3(x+6) + 4(2x-5) = 3x + 18 + 8x - 20 = 11x - 2$

   (b) $(x+3)(4x-5) = 4x^2 - 5x + 12x - 15 = 4x^2 + 7x - 15$

   (c) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{b} - \left(\sqrt{b}\right)^2 = a - b$

   Or: Use the formula for the difference of two squares to see that $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$.

   (d) $(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$.

   Note: A quicker way to expand this binomial is to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$: $(2x+3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$

   (e) See Reference Page 1 for the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using it, we get $(x+2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$.

4. (a) Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we have $4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$.

   (b) Factoring by trial and error, we get $2x^2 + 5x - 12 = (2x-3)(x+4)$.

   (c) Using factoring by grouping and the difference of two squares formula, we have $x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x^2-4)(x-3) = (x-2)(x+2)(x-3)$.

   (d) $x^4 + 27x = x(x^3 + 27) = x(x+3)(x^2 - 3x + 9)$

   This last expression was obtained using the sum of two cubes formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$. [See Reference Page 1 in the textbook.]

   (e) The smallest exponent on x is $-\frac{1}{2}$, so we will factor out $x^{-1/2}$.

   $3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x-1)(x-2)$

   (f) $x^3y - 4xy = xy(x^2 - 4) = xy(x-2)(x+2)$

$4 - x \geq 0 \quad \Rightarrow \quad x \leq 4$ and $x^2 - 1 \geq 0 \quad \Rightarrow \quad (x-1)(x+1) \geq 0 \quad \Rightarrow \quad x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

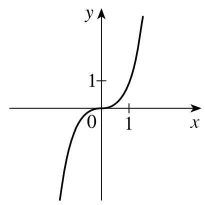
4. (a) Reflect the graph of f about the x-axis.

   (b) Stretch the graph of f vertically by a factor of 2, then shift 1 unit downward.

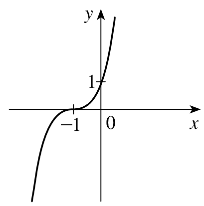
   (c) Shift the graph of f right 3 units, then up 2 units.

5. (a) Make a table and then connect the points with a smooth curve:

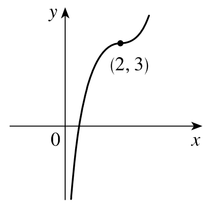
x	-2	-1	0	1	2
y	-8	-1	0	1	8



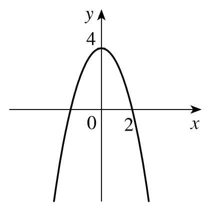
   (b) Shift the graph from part (a) left 1 unit.



   (c) Shift the graph from part (a) right 2 units and up 3 units.

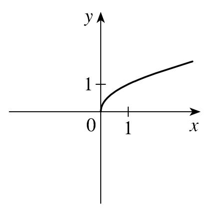


   (d) First plot $y = x^2$. Next, to get the graph of $f(x) = 4 - x^2$, reflect f about the x-axis and then shift it upward 4 units.

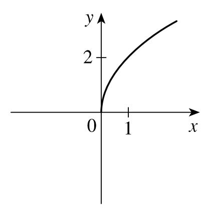


   (e) Make a table and then connect the points with a smooth curve:

x	0	1	4	9
y	0	1	2	3



   (f) Stretch the graph from part (e) vertically by a factor of two.



(c) $y = x^2(2 - x^3) = 2x^2 - x^5$ is a polynomial of degree 5.

(d) $y = \tan t - \cos t$ is a trigonometric function.

(e) $y = s/(1 + s)$ is a rational function because it is a ratio of polynomials.

(f) $y = \sqrt{x^3 - 1}/(1 + \sqrt[3]{x})$ is an algebraic function because it involves polynomials and roots of polynomials.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

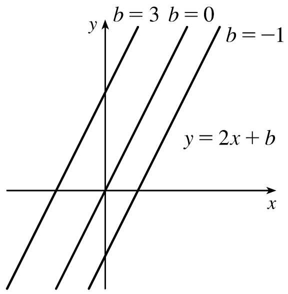
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

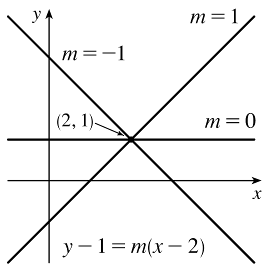
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

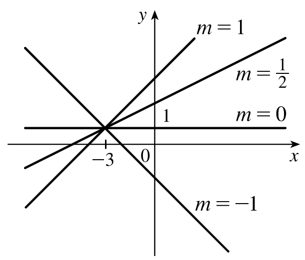


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

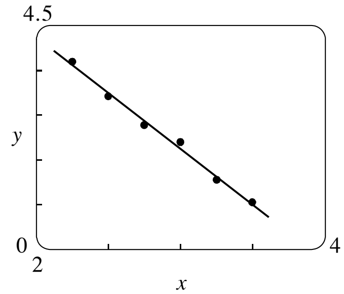


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

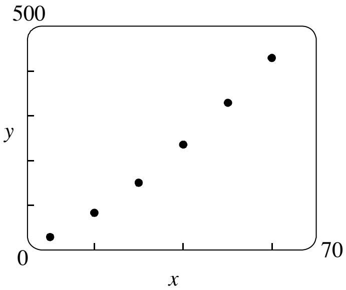
6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



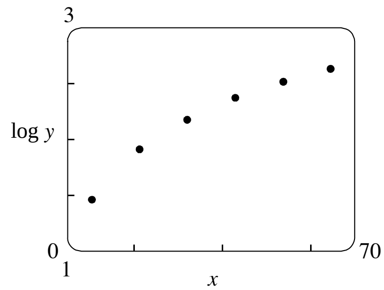
(d) Using a calculator to fit a line to the data gives $y = (-0.618857)\,x + 4.368000$.

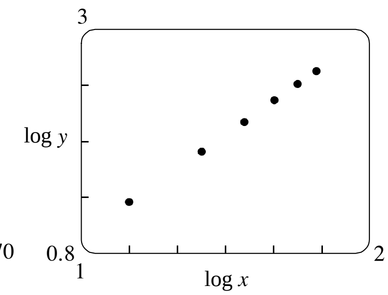


62. (a)



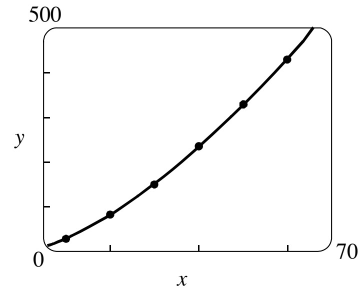
(b)



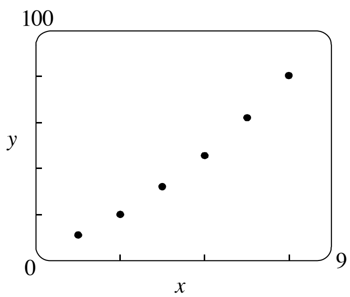


(c) Since the log-log plot is approximately linear, a power model is appropriate.

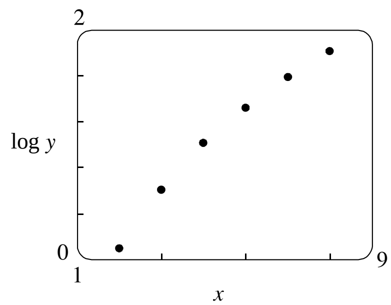
(d) Using a calculator to fit power curve to the data gives $y = (0.894488) \cdot x^{1.509230}$.

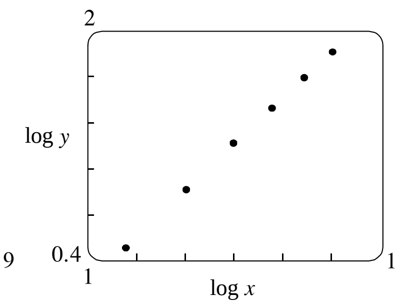


63. (a)



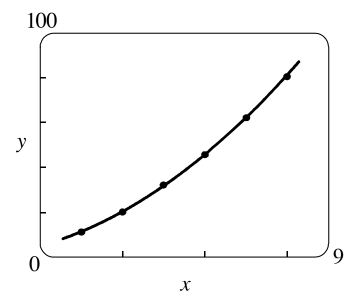
(b)





(c) Since the log-log plot is approximately linear, a power model is appropriate.

(d) Using a calculator to fit a power curve to the data gives $y = (1.260294) \cdot x^{2.002959}$



40. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

PROJECT Drug Resistance in Malaria

1. Let g_{ij} represent the frequency of ij individuals. For random and independent unions, the frequency of RR individuals will be given by the probability that any two R individuals unite, that is, $g_{\text{RR}} = p_t \cdot p_t = p_t^2$. Similarly, the probability that any two S individuals unite is $g_{\text{SS}} = (1-p_t)\cdot(1-p_t) = (1-p_t)^2$. The probability that an RS diploid individual forms is $g_{\text{RS}} = 2p_t\,(1-p_t)$. We include a factor of 2 since there are two ways of selecting RS: first an R individual, then an S individual, and vice versa. Observe that the sum of the frequencies $g_{\text{RR}} + g_{\text{RS}} + g_{\text{SS}} = 1$ as is required.

2. Scaling each frequency by the respective probabilities of survival gives $W_{\text{RR}}p_t^2$, $W_{\text{SS}}\,(1-p_t)^2$, and $2W_{\text{RS}}p_t\,(1-p_t)$. These new quantities no longer sum to one, but instead sum to $\overline{W} = W_{\text{RR}}p_t^2 + W_{\text{SS}}\,(1-p_t)^2 + 2W_{\text{RS}}p_t\,(1-p_t)$. To ensure the new diploid frequencies (after survival) g^* sum to one, we normalize or divide by $\overline{W}$ giving $g_{\text{RR}}^* = \dfrac{W_{\text{RR}}}{\overline{W}}p_t^2$, $g_{\text{SS}}^* = \dfrac{W_{\text{SS}}}{\overline{W}}\,(1-p_t)^2$, and $g_{\text{RS}}^* = 2\dfrac{W_{\text{RS}}}{\overline{W}}p_t\,(1-p_t)$.

3. If there are a total of N individuals in the population, then after reproduction, the number of R-type haploids is $bN\left(g_{\text{RR}}^* + \frac{1}{2}g_{\text{RS}}^*\right) = bN\left(\dfrac{W_{\text{RR}}}{\overline{W}}p_t^2 + \dfrac{W_{\text{RS}}}{\overline{W}}p_t\,(1-p_t)\right)$ and the total number of haploid individuals is $bN\left(g_{\text{RR}}^* + g_{\text{RS}}^* + g_{\text{SS}}^*\right) = bN\,(1) = bN$. Therefore, the frequency of R-type haploids is

$$p_{t+1} = \frac{bN\left(\dfrac{W_{\text{RR}}}{\overline{W}}p_t^2 + \dfrac{W_{\text{RS}}}{\overline{W}}p_t\,(1-p_t)\right)}{bN} = \frac{W_{\text{RR}}}{\overline{W}}p_t^2 + \frac{W_{\text{RS}}}{\overline{W}}p_t\,(1-p_t) = \frac{W_{\text{RR}}p_t^2 + W_{\text{RS}}p_t\,(1-p_t)}{W_{\text{RR}}p_t^2 + 2W_{\text{RS}}p_t\,(1-p_t) + W_{\text{SS}}\,(1-p_t)^2}.$$

4. Substituting $W_{\text{RR}} = \frac{1}{4}$ and $W_{\text{RS}} = W_{\text{SS}} = \frac{1}{2}$ into p_{t+1} gives

$$\begin{aligned} p_{t+1} &= \frac{\frac{1}{4}p_t^2 + \frac{1}{2}p_t\,(1-p_t)}{\frac{1}{4}p_t^2 + 2\cdot\frac{1}{2}p_t\,(1-p_t) + \frac{1}{2}\,(1-p_t)^2} \\ &= \frac{p_t^2 + 2p_t\,(1-p_t)}{p_t^2 + 4p_t\,(1-p_t) + 2\,(1-p_t)^2} \\ &= \frac{-p_t^2 + 2p_t}{-3p_t^2 + 4p_t + 2 - 4p_t + 2p_t^2} \\ &= \frac{p_t^2 - 2p_t}{p_t^2 - 2} \end{aligned}$$

This is the same rational function discussed in Section 1.2.

60. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta, \ldots$. Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty}\sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$

and $|\sin\theta| < 1$ $\left[\text{because } 0 < \theta < \frac{\pi}{2}\right]$.

PROJECT Modeling the Dynamics of Viral Infections

1. Viral replication is an example of exponential growth. The exponential growth recursion formula is $N(t+1) = RN(t)$ where R is the growth rate and $N(t)$ is the number of viral particles at time t. In Section 1.6, we saw the general solution of this recursion is $N_t = N_0 \cdot R^t$. With $R = 3$ and $N_0 = 1$, the recursion equation is $N_{t+1} = 3N_t$ and the general solution is $N_t = 3^t$.

2. Let t_1 be the amount of time spent in phase 1 of the infection. Solving for t_1 in the equation $N_{t_1} = N_0 \cdot R^{t_1}$ using logarithms: $\ln\left(R^{t_1}\right) = \ln\left(N_{t_1}/N_0\right) \;\Rightarrow\; t_1 = \dfrac{\ln(N_{t_1}/N_0)}{\ln(R)}$. The immune response initiates when $N_{t_1} = 2\cdot 10^6$. Therefore the time it takes for the immune response to kick in is $t_1 = \dfrac{\ln(2\cdot 10^6) - \ln(N_0)}{\ln(3)} \approx 13.2 - 0.91\ln(N_0)$. Hence, the larger the initial viral size the sooner the immune system responds.

3. Let t_2 be the amount of time since the immune response initiated, R_{immune} be the replication rate during the immune response, and d_{immune} be the number of viruses killed by the immune system at each timestep. The second phase of the infection is modeled by a two-step recursion. First, the virus replicates producing $N^* = R_{\text{immune}}N_{t_2}$ viruses. Then, the immune system kills viruses leaving $N_{t_2+1} = N^* - d_{\text{immune}}$ leftover. Combining the two steps gives the recursion formula $N_{t_2+1} = R_{\text{immune}}N_{t_2} - d_{\text{immune}}$.

4. The viral population will decrease over time if $\Delta N < 0$ at each timestep. Solving this inequality for N_{t_2}:

   $N_{t_2+1} - N_{t_2} < 0 \;\;\Rightarrow\;\; (R_{\text{immune}} - 1)N_{t_2} - d_{\text{immune}} < 0 \;\;\Rightarrow\;\; N_{t_2} < \dfrac{d_{\text{immune}}}{(R_{\text{immune}} - 1)}$ where we assumed $R_{\text{immune}} > 1$.

   Substituting the constants $R_{\text{immune}} = \frac{1}{2}\cdot 3 = 1.5$ and $d_{\text{immune}} = 500{,}000$ gives $N_{t_2} < 1{,}000{,}000$. Therefore, the immune response will cause the infection to subside over time if the viral count is less than one million. This is not possible since the immune response initiates only once the virus reaches two million copies.

5. The recursion for the third phase can be obtained from the second phase recursion formula by replacing the replication and death rates with the new values. This gives $N_{t_3+1} = R_{\text{drug}}N_{t_3} - d_{\text{drug}}$ where t_3 is the amount of time since the start of drug treatment.

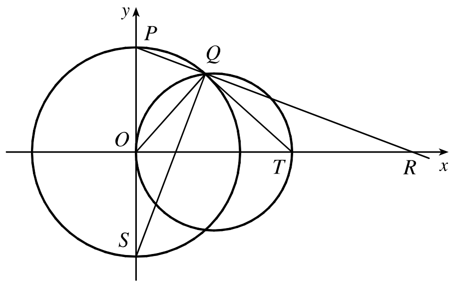
6. Similar to Problem 4, we solve for N_{t_3} in the inequality $\Delta N = N_{t_3+1} - N_{t_3} < 0$ and find that $N_{t_3} < \dfrac{d_{\text{drug}}}{(R_{\text{drug}} - 1)}$.

   Substituting the constants $R_{\text{drug}} = 1.25$ and $d_{\text{drug}} = 25{,}000{,}000$ gives $N_{t_3} < 100{,}000{,}000$. Therefore, the drug and immune system will cause the infection to subside over time if the viral count is less than 100 million. This is possible provided drug treatment begins before the viral count reaches 100 million.

Now we take the limit as $r \to 0^+$: $\lim\limits_{r\to 0^+} x = \lim\limits_{r\to 0^+} 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) = \lim\limits_{r\to 0^+} 2\left(\sqrt{1}+1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90°$ (subtended by diameter PS). So $\angle SQR = 90° = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90° - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.5 Continuity

1. From Definition 1, $\lim\limits_{x\to 4} f(x) = f(4)$.

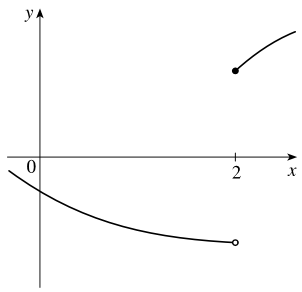
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

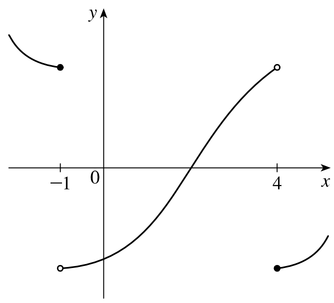
   (b) f is continuous from the left at -2 since $\lim\limits_{x\to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x\to 2^+} f(x) = f(2)$ and $\lim\limits_{x\to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4, -2)$, $(-2, 2)$, $[2, 4)$, $(4, 6)$, and $(6, 8)$.

5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim\limits_{x\to 2^+} f(x) = f(2)$.



6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim\limits_{x\to -1^-} f(x) = f(-1)$ and $\lim\limits_{x\to 4^+} f(x) = f(4)$.



Another solution: Write y as a product and make use of the Product Rule. $y = r(r^2+1)^{-1/2} \Rightarrow$
$y' = r \cdot -\frac{1}{2}(r^2+1)^{-3/2}(2r) + (r^2+1)^{-1/2} \cdot 1 = (r^2+1)^{-3/2}[-r^2 + (r^2+1)^1] = (r^2+1)^{-3/2}(1) = (r^2+1)^{-3/2}$.

The step that students usually have trouble with is factoring out $(r^2+1)^{-3/2}$. But this is no different than factoring out x^2 from $x^2 + x^5$; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

28. $y = e^{k\tan\sqrt{x}} \Rightarrow y' = e^{k\tan\sqrt{x}} \cdot \dfrac{d}{dx}\left(k\tan\sqrt{x}\right) = e^{k\tan\sqrt{x}}\left(k\sec^2\sqrt{x} \cdot \frac{1}{2}x^{-1/2}\right) = \dfrac{k\sec^2\sqrt{x}}{2\sqrt{x}}\, e^{k\tan\sqrt{x}}$

29. $y = \sin(\tan 2x) \Rightarrow y' = \cos(\tan 2x) \cdot \dfrac{d}{dx}(\tan 2x) = \cos(\tan 2x) \cdot \sec^2(2x) \cdot \dfrac{d}{dx}(2x) = 2\cos(\tan 2x)\sec^2(2x)$

30. $f(t) = \sqrt{\dfrac{t}{t^2+4}} = \left(\dfrac{t}{t^2+4}\right)^{1/2} \Rightarrow$

$$f'(t) = \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2} \cdot \frac{d}{dt}\left(\frac{t}{t^2+4}\right) = \frac{1}{2}\left(\frac{t^2+4}{t}\right)^{1/2} \cdot \frac{(t^2+4)(1) - t(2t)}{(t^2+4)^2}$$
$$= \frac{(t^2+4)^{1/2}}{2t^{1/2}} \cdot \frac{t^2+4-2t^2}{(t^2+4)^2} = \frac{4-t^2}{2t^{1/2}(t^2+4)^{3/2}}$$

31. Using Formula 5 and the Chain Rule, $y = 2^{\sin\pi x} \Rightarrow$
$y' = 2^{\sin\pi x}(\ln 2) \cdot \dfrac{d}{dx}(\sin\pi x) = 2^{\sin\pi x}(\ln 2) \cdot \cos\pi x \cdot \pi = 2^{\sin\pi x}(\pi\ln 2)\cos\pi x$

32. $y = \sin(\sin(\sin x)) \Rightarrow y' = \cos(\sin(\sin x))\,\dfrac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x))\,\cos(\sin x)\,\cos x$

33. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \Rightarrow$
$y' = 2[\cot(\sin\theta)] \cdot \dfrac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta) \cdot [-\csc^2(\sin\theta) \cdot \cos\theta] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$

34. $y = \sqrt{x + \sqrt{x + \sqrt{x}}} \Rightarrow y' = \frac{1}{2}\left(x + \sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1 + \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right]$

35. $y = \cos\sqrt{\sin(\tan\pi x)} = \cos(\sin(\tan\pi x))^{1/2} \Rightarrow$

$$y' = -\sin(\sin(\tan\pi x))^{1/2} \cdot \frac{d}{dx}(\sin(\tan\pi x))^{1/2} = -\sin(\sin(\tan\pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan\pi x))^{-1/2} \cdot \frac{d}{dx}(\sin(\tan\pi x))$$
$$= \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}} \cdot \cos(\tan\pi x) \cdot \frac{d}{dx}\tan\pi x = \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}} \cdot \cos(\tan\pi x) \cdot \sec^2(\pi x) \cdot \pi$$
$$= \frac{-\pi\cos(\tan\pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}$$

36. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\,\dfrac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

37. $y = \cos(x^2) \Rightarrow y' = -\sin(x^2) \cdot 2x = -2x\sin(x^2) \Rightarrow$
$y'' = -2x\cos(x^2) \cdot 2x + \sin(x^2) \cdot (-2) = -4x^2\cos(x^2) - 2\sin(x^2)$

38. $y = \cos^2 x = (\cos x)^2 \Rightarrow y' = 2\cos x(-\sin x) = -2\cos x\sin x \Rightarrow$
$y'' = (-2\cos x)\cos x + \sin x(2\sin x) = -2\cos^2 x + 2\sin^2 x$

Note: Many other forms of the answers exist. For example, $y' = -\sin 2x$ and $y'' = -2\cos 2x$.

21. $y = e^{\sin 2\theta} \quad \Rightarrow \quad y' = e^{\sin 2\theta} \dfrac{d}{d\theta}(\sin 2\theta) = e^{\sin 2\theta}(\cos 2\theta)(2) = 2\cos 2\theta \, e^{\sin 2\theta}$

22. $y = e^{-t}(t^2 - 2t + 2) \quad \Rightarrow$

$y' = e^{-t}(2t - 2) + (t^2 - 2t + 2)(-e^{-t}) = e^{-t}(2t - 2 - t^2 + 2t - 2) = e^{-t}(-t^2 + 4t - 4)$

23. $y = \dfrac{t}{1 - t^2} \quad \Rightarrow \quad y' = \dfrac{(1 - t^2)(1) - t(-2t)}{(1 - t^2)^2} = \dfrac{1 - t^2 + 2t^2}{(1 - t^2)^2} = \dfrac{t^2 + 1}{(1 - t^2)^2}$

24. $y = e^{mx} \cos nx \quad \Rightarrow$

$y' = e^{mx}(\cos nx)' + \cos nx \, (e^{mx})' = e^{mx}(-\sin nx \cdot n) + \cos nx \, (e^{mx} \cdot m) = e^{mx}(m \cos nx - n \sin nx)$

25. $y = \dfrac{e^{1/x}}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 (e^{1/x})' - e^{1/x} \, (x^2)'}{(x^2)^2} = \dfrac{x^2 (e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1 + 2x)}{x^4}$

26. $y = \left(\dfrac{u - 1}{u^2 + u + 1} \right)^4 \quad \Rightarrow$

$$y' = 4\left(\frac{u - 1}{u^2 + u + 1} \right)^3 \frac{d}{du}\left(\frac{u - 1}{u^2 + u + 1} \right) = 4\left(\frac{u - 1}{u^2 + u + 1} \right)^3 \frac{(u^2 + u + 1)(1) - (u - 1)(2u + 1)}{(u^2 + u + 1)^2}$$

$$= \frac{4(u - 1)^3}{(u^2 + u + 1)^3} \frac{u^2 + u + 1 - 2u^2 + u + 1}{(u^2 + u + 1)^2} = \frac{4(u - 1)^3(-u^2 + 2u + 2)}{(u^2 + u + 1)^5}$$

27. $\dfrac{d}{dx}(xy^4 + x^2 y) = \dfrac{d}{dx}(x + 3y) \quad \Rightarrow \quad x \cdot 4y^3 y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \quad \Rightarrow$

$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \quad \Rightarrow \quad y' = \dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

28. $y = \ln(\csc 5x) \quad \Rightarrow \quad y' = \dfrac{1}{\csc 5x}(-\csc 5x \, \cot 5x)(5) = -5 \cot 5x$

29. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \quad \Rightarrow$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \, \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta \, [(1 + \tan 2\theta)\tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2\sec 2\theta \, (\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta \, (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \qquad [1 + \tan^2 x = \sec^2 x]$$

30. $\dfrac{d}{dx}(x^2 \cos y + \sin 2y) = \dfrac{d}{dx}(xy) \quad \Rightarrow \quad x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \quad \Rightarrow$

$y'(-x^2 \sin y + 2\cos 2y - x) = y - 2x \cos y \quad \Rightarrow \quad y' = \dfrac{y - 2x \cos y}{2\cos 2y - x^2 \sin y - x}$

31. $y = e^{cx}(c \sin x - \cos x) \quad \Rightarrow$

$y' = e^{cx}(c \cos x + \sin x) + ce^{cx}(c \sin x - \cos x) = e^{cx}(c^2 \sin x - c \cos x + c \cos x + \sin x)$

$= e^{cx}(c^2 \sin x + \sin x) = e^{cx} \sin x \, (c^2 + 1)$

32. $y = \ln(x^2 e^x) = \ln x^2 + \ln e^x = 2 \ln x + x \quad \Rightarrow \quad y' = 2/x + 1$

33. $y = \log_5(1 + 2x) \quad \Rightarrow \quad y' = \dfrac{1}{(1 + 2x) \, \ln 5} \dfrac{d}{dx}(1 + 2x) = \dfrac{2}{(1 + 2x) \, \ln 5}$

3. This limit has the form $\frac{0}{0}$. $\lim_{x\to(\pi/2)^+} \frac{\cos x}{1-\sin x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^+} \frac{-\sin x}{-\cos x} = \lim_{x\to(\pi/2)^+} \tan x = -\infty.$

4. This limit has the form $\frac{0}{0}$. $\lim_{x\to0} \frac{\sin 4x}{\tan 5x} \overset{\text{H}}{=} \lim_{x\to0} \frac{4\cos 4x}{5\sec^2(5x)} = \frac{4(1)}{5(1)^2} = \frac{4}{5}$

5. This limit has the form $\frac{0}{0}$. $\lim_{t\to0} \frac{e^t-1}{t^3} \overset{\text{H}}{=} \lim_{t\to0} \frac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

6. This limit has the form $\frac{0}{0}$. $\lim_{t\to0} \frac{e^{3t}-1}{t} \overset{\text{H}}{=} \lim_{t\to0} \frac{3e^{3t}}{1} = 3$

7. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

8. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{\csc\theta} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

9. $\lim_{x\to0^+} [(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x \to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

10. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} 2\lim_{x\to\infty} \frac{1/x}{1} = 2(0) = 0$

11. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} \overset{\text{H}}{=} \lim_{x\to0} \frac{\frac{1}{2}(1+2x)^{-1/2}\cdot 2 - \frac{1}{2}(1-4x)^{-1/2}(-4)}{1}$$
$$= \lim_{x\to0}\left(\frac{1}{\sqrt{1+2x}} + \frac{2}{\sqrt{1-4x}}\right) = \frac{1}{\sqrt{1}} + \frac{2}{\sqrt{1}} = 3$$

12. This limit has the form $\frac{0}{0}$. $\lim_{x\to1} \frac{\ln x}{\sin \pi x} \overset{\text{H}}{=} \lim_{x\to1} \frac{1/x}{\pi\cos\pi x} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$

13. This limit has the form $\frac{0}{0}$. $\lim_{t\to0} \frac{5^t-3^t}{t} \overset{\text{H}}{=} \lim_{t\to0} \frac{5^t\ln 5 - 3^t \ln 3}{1} = \ln 5 - \ln 3 = \ln\frac{5}{3}$

14. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{3u^2} \overset{\text{H}}{=} \frac{1}{30}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\text{H}}{=} \frac{1}{600}\lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{1} = \frac{1}{6000}\lim_{u\to\infty} e^{u/10} = \infty$$

15. This limit has the form $\frac{0}{0}$. $\lim_{x\to0} \frac{e^x-1-x}{x^2} \overset{\text{H}}{=} \lim_{x\to0} \frac{e^x-1}{2x} \overset{\text{H}}{=} \lim_{x\to0} \frac{e^x}{2} = \frac{1}{2}$

16. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to0} \frac{\cos mx - \cos nx}{x^2} \overset{\text{H}}{=} \lim_{x\to0} \frac{-m\sin mx + n\sin nx}{2x} \overset{\text{H}}{=} \lim_{x\to0} \frac{-m^2\cos mx + n^2\cos nx}{2} = \frac{1}{2}\left(n^2-m^2\right)$$

17. This limit has the form $\frac{0}{0}$. $\lim_{x\to1} \frac{1-x+\ln x}{1+\cos\pi x} \overset{\text{H}}{=} \lim_{x\to1} \frac{-1+1/x}{-\pi\sin\pi x} \overset{\text{H}}{=} \lim_{x\to1} \frac{-1/x^2}{-\pi^2\cos\pi x} = \frac{-1}{-\pi^2(-1)} = -\frac{1}{\pi^2}$

(b) $\dfrac{dp}{dt} = g(\hat{p}) = s\hat{p}\,(1-\hat{p}) = 0 \quad\Rightarrow\quad \hat{p} = 0$ or $\hat{p} = 1$. Also $g'(p) = s - 2sp$, so $g'(0) = s$ and $g'(1) = s - 2s = -s$. Therefore, $\hat{p} = 0$ is locally stable when $s < 0$ and $\hat{p} = 1$ is locally stable when $s > 0$.

17. We know that r_1 is a linear function of $1 - p$ (frequency of type 2) passing through the points $(1-p, r_1) = (0,0)$ and $(1,\alpha)$, so the growth rate of type 1 is $r_1 = \dfrac{\alpha - 0}{1 - 0}(1 - p - 0) + 0 = \alpha(1-p)$. Similarly, r_2 is a linear function of p (frequency of type 1) passing through the points $(p, r_2) = (0,0)$ and $(1,\beta)$, so the growth rate of type 2 is $r_2 = \dfrac{\beta - 0}{1 - 0}(p - 0) + 0 = \beta p$. Thus, the growth of the two bacterial strains is modelled by $dN_1/dt = r_1 N_1 = \alpha(1-p)N_1$ and $dN_2/dt = r_2 N_2 = \beta p N_2$. As in Exercise 16a, we have

$$\frac{dp}{dt} = \frac{N_1'\,(N_1+N_2) - N_1\,(N_1'+N_2')}{(N_1+N_2)^2} = \frac{N_1'N_2 - N_1N_2'}{(N_1+N_2)^2} = \frac{(\alpha(1-p)N_1)\,N_2 - N_1\,(\beta p N_2)}{(N_1+N_2)^2}$$

$$= \frac{N_1N_2\,(\alpha(1-p) - \beta p)}{(N_1+N_2)^2} = \left(\frac{N_1}{N_1+N_2}\right)\left(\frac{N_2}{N_1+N_2}\right)[\alpha(1-p) - \beta p]$$

$$= \left(\frac{N_1}{N_1+N_2}\right)\left(1 - \frac{N_1}{N_1+N_2}\right)[\alpha(1-p) - \beta p] = p\,(1-p)\,[\alpha(1-p) - \beta p]$$

This is the differential equation for the cross-feeding model used in Example 7.

18. (a) $\dfrac{d\hat{p}}{dt} = g(\hat{p}) = \hat{p}(1-\hat{p})[\alpha(1-\hat{p}) - \beta\hat{p}] = 0 \quad\Rightarrow\quad \hat{p} = 0 \quad \text{and} \quad \hat{p} = 1 \quad \text{and} \quad \alpha(1-\hat{p}) - \beta\hat{p} = 0 \quad\Rightarrow\quad \hat{p} = \dfrac{\alpha}{\alpha+\beta}$.

(b) $g'(p) = (1-2p)[\alpha(1-p) - \beta p] + p(1-p)[-\alpha - \beta]$

So $g'(0) = \alpha > 0 \quad\Rightarrow\quad \hat{p} = 0$ is an unstable equilibrium

and $g'(1) = (-1)\,(-\beta) = \beta > 0 \quad\Rightarrow\quad \hat{p} = 1$ is an unstable equilibrium

and $g'\left(\dfrac{\alpha}{\alpha+\beta}\right) = \left(1 - \dfrac{2\alpha}{\alpha+\beta}\right)\left[\alpha\dfrac{\beta}{\alpha+\beta} - \beta\dfrac{\alpha}{\alpha+\beta}\right] + \left(\dfrac{\alpha}{\alpha+\beta}\right)\left(\dfrac{\beta}{\alpha+\beta}\right)[-\alpha-\beta] = -\dfrac{\alpha\beta\,(\alpha+\beta)}{(\alpha+\beta)^2}$

$$= -\frac{\alpha\beta}{\alpha+\beta} < 0 \quad\Rightarrow\quad \hat{p} = \frac{\alpha}{\alpha+\beta} \text{ is locally stable}$$

19. (a) With a type 1 to type 2 per capita mutation rate μ, the rate of change of the two bacterial strains are $dN_1/dt = r_1N_1 - \mu N_1 = (r_1 - \mu)\,N_1$ and $dN_2/dt = r_2N_2 + \mu N_1$. So $p(t) = \dfrac{N_1(t)}{N_1(t) + N_2(t)} \quad\Rightarrow$

$$\frac{dp}{dt} = \frac{N_1'\,(N_1+N_2) - N_1\,(N_1'+N_2')}{(N_1+N_2)^2} = \frac{N_1'N_2 - N_1N_2'}{(N_1+N_2)^2} = \frac{[(r_1-\mu)\,N_1]\,N_2 - N_1\,(r_2N_2 + \mu N_1)}{(N_1+N_2)^2}$$

$$= \frac{(r_1 - \mu - r_2)\,N_1N_2 - \mu N_1^2}{(N_1+N_2)^2} = (r_1 - \mu - r_2)\left(\frac{N_1}{N_1+N_2}\right)\left(\frac{N_2}{N_1+N_2}\right) - \mu\left(\frac{N_1}{N_1+N_2}\right)^2$$

$$= (r_1 - \mu - r_2)\left(\frac{N_1}{N_1+N_2}\right)\left(1 - \frac{N_1}{N_1+N_2}\right) - \mu\left(\frac{N_1}{N_1+N_2}\right)^2 = (r_1 - \mu - r_2)\,p\,(1-p) - \mu p^2$$

$$= (r_1 - r_2)\,p\,(1-p) - \mu p\,(1-p) - \mu p^2 = (r_1 - r_2)\,p\,(1-p) - \mu p$$

$$= sp\,(1-p) - \mu p \quad \text{where } s = r_1 - r_2$$

(b) $\dfrac{d\hat{p}}{dt} = g(\hat{p}) = s\hat{p}\,(1-\hat{p}) - \mu\hat{p} = 0 \quad\Rightarrow\quad \hat{p}\,[s\,(1-\hat{p}) - \mu] = 0 \quad\Rightarrow\quad \hat{p} = 0 \quad \text{and} \quad s\,(1-\hat{p}) - \mu = 0 \quad\Rightarrow$ $\hat{p} = 1 - \dfrac{\mu}{s}$

(c) $g'(p) = s - 2sp - \mu \quad\Rightarrow\quad g'(0) = s - \mu$. So $\hat{p} = 0$ is locally stable when $s < \mu$, that is when $r_1 < r_2 + \mu$. Also, $g'\left(1 - \dfrac{\mu}{s}\right) = s - 2s\left(1 - \dfrac{\mu}{s}\right) - \mu = -s + \mu$, so $\hat{p} = 1 - \dfrac{\mu}{s}$ is locally stable when $s > \mu$, that is when $r_1 > r_2 + \mu$.

11. $\dfrac{dr}{dt}+2tr=r \;\Rightarrow\; \dfrac{dr}{dt}=r-2tr=r(1-2t) \;\Rightarrow\; \displaystyle\int\frac{dr}{r}=\int(1-2t)\,dt$ [if $r\neq 0$] $\Rightarrow\; \ln|r|=t-t^2+C \;\Rightarrow$ $|r|=e^{t-t^2+C}=ke^{t-t^2}$. (Note that $r=0$ is also a solution but it does not satisfy the initial condition.) Since $r(0)=5$, $5=ke^0=k$. Thus, $r(t)=5e^{t-t^2}$.

12. $(1+\cos x)y'=(1+e^{-y})\sin x \;\Rightarrow\; \dfrac{dy}{1+e^{-y}}=\dfrac{\sin x\,dx}{1+\cos x} \;\Rightarrow\; \displaystyle\int\frac{dy}{1+1/e^y}=\int\frac{\sin x\,dx}{1+\cos x} \;\Rightarrow$

$\displaystyle\int\frac{e^y\,dy}{1+e^y}=\int\frac{\sin x\,dx}{1+\cos x} \;\Rightarrow\; \ln|1+e^y|=-\ln|1+\cos x|+C \;\Rightarrow\; \ln(1+e^y)=-\ln(1+\cos x)+C \;\Rightarrow$

$1+e^y=e^{-\ln(1+\cos x)}\cdot e^C \;\Rightarrow\; e^y=ke^{-\ln(1+\cos x)}-1 \;\Rightarrow\; y=\ln[ke^{-\ln(1+\cos x)}-1]$. Since $y(0)=0$, $0=\ln[ke^{-\ln 2}-1] \;\Rightarrow\; e^0=k\left(\frac{1}{2}\right)-1 \;\Rightarrow\; k=4$. Thus, $y(x)=\ln[4e^{-\ln(1+\cos x)}-1]$. An equivalent form is $y(x)=\ln\dfrac{3-\cos x}{1+\cos x}$.

13. $\dfrac{dn}{dt}=\cos\left(\dfrac{2\pi t}{365}\right)n \;\Rightarrow\; \displaystyle\int\frac{dn}{n}=\int\cos\left(\frac{2\pi t}{365}\right)dt$ [if $n\neq 0$] $\Rightarrow\; \ln|n|=\dfrac{365}{2\pi}\sin\left(\dfrac{2\pi t}{365}\right)+C \;\Rightarrow$ $|n|=e^Ce^{(365/2\pi)\sin(2\pi t/365)} \;\Rightarrow\; n=Ae^{(365/2\pi)\sin(2\pi t/365)}$ where $A=\pm e^C$ is a constant. Note that $n=0$ is also a solution and this can be included in the family of solutions by allowing A to be zero. Now $n(0)=n_0$, so $A=n_0$. Therefore, the population size after t days is $n(t)=n_0e^{(365/2\pi)\sin(2\pi t/365)}$.

14. $\dfrac{dn}{dt}=r\left(\cos\left[\dfrac{2\pi t}{365}\right]-at\right)n \;\Rightarrow\; \displaystyle\int\frac{dn}{n}=r\int\left(\cos\left[\frac{2\pi t}{365}\right]-at\right)dt$ [if $n\neq 0$] $\Rightarrow$ $\ln|n|=r\left(\dfrac{365}{2\pi}\sin\left[\dfrac{2\pi t}{365}\right]-\frac{1}{2}at^2\right)+C \;\Rightarrow\; n=Ae^{r\left((365/2\pi)\sin[2\pi t/365]-at^2/2\right)}$ where $A=\pm e^C$ is a constant. Note that $n=0$ is also a solution and this can be included in the family of solutions by allowing A to be zero. Now $n(0)=n_0$, so $A=n_0$. Therefore, the population size after t days is $n(t)=n_0e^{r\left((365/2\pi)\sin[2\pi t/365]-at^2/2\right)}$.

15. (a) $\dfrac{dp}{dt}=cp(1-p)-mp \;\Rightarrow\; \displaystyle\int\frac{dp}{cp(1-p)-mp}=\int dt$ $\begin{bmatrix}\text{if } p\neq 0 \text{ and}\\ p\neq 1-m/c\end{bmatrix}$ $\Rightarrow\; \displaystyle\int\frac{dp}{p\,(c-m-cp)}=t+C_1$

We can evaluate the integral by writing the partial fraction decomposition of the integrand, provided $c\neq m$. This gives $\dfrac{1}{p\,(c-m-cp)}=\dfrac{A}{p}+\dfrac{B}{c-m-cp} \;\Leftrightarrow\; 1=A\,(c-m-cp)+Bp \;\Leftrightarrow\; 1=(B-Ac)\,p+A\,(c-m)$. Setting $p=0$ gives $1=A\,(c-m)$, so $A=1/(c-m)$. Equating coefficients of p gives $B-Ac=0$, so $B=Ac=c/(c-m)$.

Therefore, $\displaystyle\int\frac{dp}{p\,(c-m-cp)}=\int\left(\frac{1/(c-m)}{p}+\frac{c/(c-m)}{c-m-cp}\right)dp=\frac{1}{c-m}\int\left(\frac{1}{p}+\frac{c}{c-m-cp}\right)dp$

$$=\frac{1}{c-m}\left(\ln|p|-\ln|c-m-cp|\right)=\frac{1}{c-m}\ln\left|\frac{p}{c-m-cp}\right|$$

Continuing to solve the differential equation, we have $\dfrac{1}{c-m}\ln\left|\dfrac{p}{c-m-cp}\right|=t+C_1 \;\Leftrightarrow$

$\ln\left|\dfrac{p}{c-m-cp}\right|=(c-m)\,t+(c-m)\,C_1 \;\Leftrightarrow\; \left|\dfrac{p}{c-m-cp}\right|=e^{(c-m)t+(c-m)C_1} \;\Leftrightarrow$

$\dfrac{p}{c-m-cp}=C_2e^{(c-m)t}$ where $C_2=\pm e^{(c-m)C_1} \;\Leftrightarrow\; p=C_2\,(c-m)\,e^{(c-m)t}-C_2ce^{(c-m)t}p \;\Leftrightarrow$

[continued]

24. The equilibria of the system $\left\{\begin{array}{l} x' = f_1(x,y) = -xy + y + ax \\ y' = f_2(x,y) = 2y - xy \end{array}\right\}$ must satisfy $\left\{\begin{array}{l} 0 = -xy + y + ax \\ 0 = y(2-x) \end{array}\right\}$. Thus, the equilibria are (i) $\hat{x} = 0$, $\hat{y} = 0$ and (ii) $\hat{x} = 2$, $\hat{y} = 2a$. The Jacobian matrix of the differential system is

$$J(x,y) = \begin{bmatrix} \dfrac{\partial f_1(x,y)}{\partial x} & \dfrac{\partial f_1(x,y)}{\partial y} \\ \dfrac{\partial f_2(x,y)}{\partial x} & \dfrac{\partial f_2(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} -y+a & -x+1 \\ -y & 2-x \end{bmatrix} \quad \Rightarrow \quad J(0,0) = \begin{bmatrix} a & 1 \\ 0 & 2 \end{bmatrix} \text{ and } J(2,2a) = \begin{bmatrix} -a & -1 \\ -2a & 0 \end{bmatrix}.$$

The eigenvalues of $J(0,0)$ are $\lambda_1 = a$ and $\lambda_2 = 2$, so by Theorem 14, equilibrium (i) is unstable. Also, $\det J(2,2a) = -2a$ and trace $J(2,2a) = -a$ so the determinant and trace have the same sign if $a \neq 0$. Therefore, equilibrium (ii) is unstable by Theorem 16, however, if $a = 0$ the stability analysis is inconclusive.

25. The equilibria of the system $\left\{\begin{array}{l} x' = f_1(x,y) = ax^2 + ay - x \\ y' = f_2(x,y) = x - y \end{array}\right\}$ must satisfy $\left\{\begin{array}{l} 0 = ax^2 + ay - x \\ 0 = x - y \end{array}\right\}$. The second equation specifies that $y = x$. Substituting this into the first equation gives $ax^2 + ax - x = 0 \quad \Rightarrow \quad x(ax + a - 1) = 0$. Thus, the equilibria are (i) $\hat{x} = 0$, $\hat{y} = 0$ and (ii) $\hat{x} = \dfrac{1-a}{a}$, $\hat{y} = \dfrac{1-a}{a}$. The Jacobian matrix of the differential system is

$$J(x,y) = \begin{bmatrix} \dfrac{\partial f_1(x,y)}{\partial x} & \dfrac{\partial f_1(x,y)}{\partial y} \\ \dfrac{\partial f_2(x,y)}{\partial x} & \dfrac{\partial f_2(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 2ax-1 & a \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad J(0,0) = \begin{bmatrix} -1 & a \\ 1 & -1 \end{bmatrix} \text{ and}$$

$J\left(\dfrac{1-a}{a}, \dfrac{1-a}{a}\right) = \begin{bmatrix} 1-2a & a \\ 1 & -1 \end{bmatrix}$. Now $\det J(0,0) = 1 - a$ and trace $J(0,0) = -2 < 0$, so by Theorem 16, equilibrium (i) is locally stable when $a < 1$. Also, $\det J\left(\dfrac{1-a}{a}, \dfrac{1-a}{a}\right) = a - 1$ and trace $J\left(\dfrac{1-a}{a}, \dfrac{1-a}{a}\right) = -2a$, so equilibrium (ii) is locally stable when $a > 1$. Note that the stability analysis is inconclusive for both equilibria when $a = 1$.

26. (a) $\dfrac{dM}{dt} = f_1(M,C) = 2C + CM^2 - \dfrac{10M}{1+M} \qquad \dfrac{dC}{dt} = f_2(M,C) = 1 - M \qquad [\alpha = 2, \beta = 1, \gamma = 10, \delta = 1]$

The equilibria must satisfy $\left\{\begin{array}{r} 2C + CM^2 - \dfrac{10M}{1+M} = 0 \\ 1 - M = 0 \end{array}\right\}$. The second equation requires that $M = 1$ and substituting this into the first equation gives $2C + C - 5 = 0 \quad \Rightarrow \quad C = \frac{5}{3}$. Thus the only equilibrium is $\hat{M} = 1$, $\hat{C} = \frac{5}{3}$.

(b) $J(M,C) = \begin{bmatrix} \dfrac{\partial f_1(M,C)}{\partial M} & \dfrac{\partial f_1(M,C)}{\partial C} \\ \dfrac{\partial f_2(M,C)}{\partial M} & \dfrac{\partial f_2(M,C)}{\partial C} \end{bmatrix}$

$$= \begin{bmatrix} 2CM - \dfrac{10(1+M) - 10M}{(1+M)^2} & 2 + M^2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2CM - \dfrac{10}{(1+M)^2} & 2 + M^2 \\ -1 & 0 \end{bmatrix}$$

(c) $J\left(1, \frac{5}{3}\right) = \begin{bmatrix} \frac{5}{6} & 3 \\ -1 & 0 \end{bmatrix} \quad \Rightarrow \quad \det J\left(1, \frac{5}{3}\right) = 3 > 0$ and trace $J\left(1, \frac{5}{3}\right) = \frac{5}{6} > 0$. Therefore, by Theorem 16, the equilibrium is unstable.

60. (a) Let X be a binomial random variable in which 'recovered' is treated as a "success". We are told that $p = 1 - 0.6 = 0.4$, and there are $n = 10$ patients, so the probability that 8 or more patients would recover (without the drug) is

$$\begin{aligned} P(X \geq 8) &= P(X = 8) + P(X = 9) + P(X = 10) \\ &= \binom{10}{8}(0.4)^8(0.6)^2 + \binom{10}{9}(0.4)^9(0.6)^1 + \binom{10}{10}(0.4)^{10}(0.6)^0 \\ &= 45(0.4)^8(0.6)^2 + 10(0.4)^9(0.6) + (0.4)^{10} \\ &= 0.01229 \end{aligned}$$

(b) The probability found in part (a) is less than the 0.05 threshold. This indicates that without drug intervention it is highly unlikely for 8 or more patients to recover from the disease. Since 8 out of the 10 drug recipients recovered, the drug treatment appears to be effective.

61. We can treat the number of NK cells X as a binomial random variable with $p = 0.07$ and $n = 10$ assuming that the outcome of each trial is independent.

(a) $P(X = 0) = \binom{10}{0}(0.07)^0(0.93)^{10} = (0.93)^{10} \approx 0.484$

(b) $P(X = 2) = \binom{10}{2}(0.07)^2(0.93)^8 = 45(0.07)^2(0.93)^8 \approx 0.123$

(c) $$\begin{aligned} P(X \leq 3) &= P(0) + P(1) + P(2) + P(3) \\ &= \binom{10}{0}(0.07)^0(0.93)^{10} + \binom{10}{1}(0.07)^1(0.93)^9 + \binom{10}{2}(0.07)^2(0.93)^8 + \binom{10}{3}(0.07)^3(0.93)^7 \\ &= (0.93)^{10} + 10(0.07)(0.93)^9 + 45(0.07)^2(0.93)^8 + 120(0.07)^3(0.93)^7 \\ &\approx 0.996 \end{aligned}$$

62. We can treat the number of GC nucleotides X as a binomial random variable with $p = 0.3$ and $n = 10$ assuming that each GC nucleotide identity is independent of one another.

(a) $P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7 = 120(0.3)^3(0.7)^7 \approx 0.2668$

(b) $P(X > 8) = P(X = 9) + P(X = 10) = \binom{10}{9}(0.3)^9(0.7)^1 + \binom{10}{10}(0.3)^{10}(0.7)^0 = 10(0.3)^9(0.7) + (0.3)^{10} \approx 0.00014$

63. We can treat the number of GC nucleotides X as a binomial random variable with $n = 12$ and $i = 3$ assuming that each GC nucleotide identity is independent of one another.

(a) $p = 0.5 \quad \Rightarrow \quad P(X = 3) = \binom{12}{3}(0.5)^3(0.5)^9 = 220(0.5)^{12} \approx 0.0537$

(b) $p = 0.3 \quad \Rightarrow \quad P(X = 3) = \binom{12}{3}(0.3)^3(0.7)^9 = 220(0.3)^3(0.7)^9 \approx 0.2397$

(c) If the GC content of the virus is p, then using Definition 16 with $i = 3$ and $n = 12$ gives

$$P(X = 3) = \binom{12}{3}p^3(1-p)^9 = 220p^3(1-p)^9$$

(d) The maximum value of $P(X = 3)$ must satisfy $\frac{d}{dp}\left[220p^3(1-p)^9\right] = 0 \quad \Rightarrow \quad 220\left[3p^2(1-p)^9 - 9p^3(1-p)^8\right] = 0$ $\Rightarrow \quad p^2(1-p)^8[(1-p) - 3p] = 0 \quad \Rightarrow \quad p^2(1-p)^8[1-4p] = 0 \quad \Rightarrow \quad p = 0, 1, \frac{1}{4}$. Now, $P(X = 3) = 0$ when $p = 0$ and $p = 1$. Also, when $p = \frac{1}{4}$, $P(X = 3) = 220\left(\frac{1}{4}\right)^3\left(1 - \frac{1}{4}\right)^9 \approx 0.258$. Thus, by using the Closed Interval Method on the domain $[0, 1]$, we have found that the absolute maximum value of $P(X = 3)$ occurs when $p = 1/4$.