Contents

Problems marked with * are more difficult.

Problems marked with † are substantially affected by typos which are included in the Errata.

Chapter 1

Solutions to Chapter 1 Exercises

Exercise 1.1. According to the prescription of the text, we assume that the solutions in the two regions take the form

$$\Psi_1(x) = A_1 \sin(kx) + B_1 \cos(kx) \ , \tag{1}$$

and

$$\Psi_2(x) = A_2 e^{\kappa x} + B_2 e^{-\kappa x} \ . \tag{2}$$

There is no need to write down the solution for $\Psi_3(x)$, since the complete solution is either symmetric, in which case the derivative of $\Psi_2(x)$ vanishes at $x = 0$, or antisymmetric, therefore, $\Psi_2(x)$ itself vanishes at $x = 0$.

The boundary conditions of the problem are

$$\Psi_1\left(x = -a - \frac{b}{2}\right) = 0 \longrightarrow A_1 \sin\left(k\left(-a - \frac{b}{2}\right)\right) + B_1 \cos\left(k\left(-a - \frac{b}{2}\right)\right) = 0 \ . \tag{3}$$

$$\Psi_1\left(x = -\frac{b}{2}\right) = \Psi_2\left(x = -\frac{b}{2}\right) \longrightarrow A_1 \sin\left(k\left(-\frac{b}{2}\right)\right) + B_1 \cos\left(k\left(-\frac{b}{2}\right)\right) = A_2 e^{-\kappa \frac{b}{2}} + B_2 e^{\kappa \frac{b}{2}} \tag{4}$$

$$\frac{\Psi_1}{dx}\bigg|_{x=-\frac{b}{2}} = \frac{\Psi_2}{dx}\bigg|_{x=-\frac{b}{2}} \longrightarrow A_1 k \cos\left(k\left(-\frac{b}{2}\right)\right) - B_1 k \sin\left(k\left(-\frac{b}{2}\right)\right) = A_2 \kappa e^{\kappa \frac{b}{2}} - B_2 \kappa e^{-\kappa \frac{b}{2}} \ , \tag{5}$$

and finally,

$$\Psi_2(x = 0) = 0 \longrightarrow A_2 + B_2 = 0 \tag{6}$$

i.e., $A_2 = -B_2$ for the antisymmetric case, or

$$\frac{\Psi_2}{dx}\bigg|_{x=0} = 0 \longrightarrow A_2 \kappa - B_2 \kappa = 0 \ , \tag{7}$$

i.e., $A_2 = B_2$ for the symmetric case. Regarding the amplitudes A_1, B_1, A_2 and B_2 as the unknowns, we get a homogeneous set of four linear equations, which means that in order to have a solution,

are

$$
\begin{aligned}
|\Psi_1\rangle &= |p_x\rangle + |p_y\rangle + |p_z\rangle \\
|\Psi_2\rangle &= |p_x\rangle - |p_y\rangle - |p_z\rangle \\
|\Psi_3\rangle &= -|p_x\rangle + |p_y\rangle - |p_z\rangle \\
|\Psi_4\rangle &= -|p_x\rangle - |p_y\rangle + |p_z\rangle
\end{aligned}
\tag{1}
$$

However, the scalar product of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is

$$
\langle\Psi_1|\Psi_2\rangle = 1 - 1 - 1 = -1 \ .
\tag{2}
$$

Exercise 1.27. The density of states in the presence of disorder is given by Eq. (1.72), where we can take $D(E-E_g) = \sqrt{E - E_g}$, i.e., we can drop the irrelevant constants in front of the square root. When performing the integral, we need to pay attention to the argument of the square root, namely, it should not be negative. Therefore, the integral in Eq. (1.72) is to be taken between $-\infty$ and E. This is physically sound, since for energies smaller than the gap, the contribution of a particular band is zero. Thus the integral we are to calculate is

$$
D(E) = \frac{1}{\sqrt{2\pi}\Delta E} \int_{-\infty}^{E} dE_g \ \sqrt{E - E_g}\, e^{-(E_g(0)-E_g)^2/2\Delta E^2} \ .
\tag{1}
$$

These densities of states are shown in three representative cases, with $E_g(0) = 1.5$. As it can be seen in Fig. 1.13, when $\Delta E \to 0$, we recover the well-known 3D result.
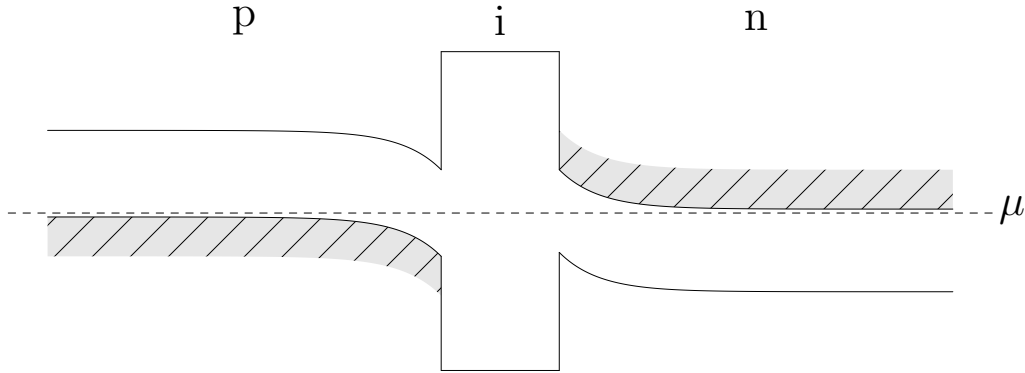
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p   i   n

Figure 2.2: Band bending in a p-i-n structure.

Exercise 2.11.   By the definition of the density of states, in 1D we have

$$D(E)dE = \frac{L}{2\pi}dk \ , \tag{1}$$

i.e.,

$$D(E) = \frac{L}{2\pi}\frac{dk}{dE} = \frac{L}{2\pi}\left(\frac{dE}{dk}\right)^{-1} = \frac{L}{2\pi}\left(\frac{\hbar^2 k}{m}\right)^{-1} = \frac{L}{2\pi}\frac{m\hbar}{\hbar^2\sqrt{2mE}} = \frac{L}{2\pi}\frac{\sqrt{m}}{\hbar\sqrt{2}\sqrt{E}} \ . \tag{2}$$

Exercise 2.12.   In order to calculate the number of particles for the quantum well, we first have to determine how many quantum levels are occupied, for which we would need the confinement energies. This is given by

$$E_\ell = \frac{\hbar^2\pi^2}{mL^2}\ell^2 = \frac{\left(2\cdot 10^{-5}\ \text{eV}\cdot\text{s}\right)^2 \times 3.14^2}{5.1\cdot 10^5\ \text{eV} \times \left(10^{-6}\ \text{cm}\right)^2}\ell^2 = \ell^2\cdot 7.8\ \text{meV} \ , \tag{1}$$

from which we can conclude that only the first, second and third confined levels will be occupied. The energies of these states are equal to 7.8, 31 and 70 meV, respectively. This means that the Fermi energies will be 100 meV, 69 meV, and 30 meV, respectively.

Since for a 2D system the density of states is given by

$$D_{2d}(E) = \frac{A}{2\pi}\frac{m}{\hbar^2} \ , \tag{2}$$

the number of particles is equal to

$$N = \int_0^{E_F} D(E)\,dE = \frac{A}{2\pi}\frac{m}{\hbar^2}E_F \ , \tag{3}$$

$u(x = L) = 1200$ J/cm^3, $C_1 = \dfrac{GL}{2D}$. Clearly, since the temperature (and thus the energy density) are kept at the same value at the two ends, the maximum of the temperature distribution will be at $L/2$. The temperature is obtained from $T(x) = u(x)/c\varrho$, i.e.,

$$T(x) = -\frac{G}{2Dc\varrho}x^2 + \frac{GL}{2Dc\varrho}x + 300 \text{ K} , \tag{3}$$

with the maximum being equal to

$$T_{max} = T(L/2) = 300 + \frac{GL^2}{8Dc\varrho} = 300 + \frac{GL^2}{8K} = 300 + \frac{20 \text{ W/cm}^3 \times (100 \text{ cm})^2}{8 \times 1 \text{ W/(K} \cdot \text{cm)}} = 25300 \text{ K} . \tag{4}$$

This temperature is far beyond the melting point of any material, thus, any real bar would melt, if it was exposed to such a heat load.

Upon symmetry, the heat flow at the ends of the bar will be equal, so we calculate it at $x = 0$:

$$h(x = 0) = -D\frac{du}{dx}\bigg|_{x=0} = -\frac{GL}{2} . \tag{5}$$

The total heat flow at the ends is thus equal to $h_{tot} = h(x = 0) + h(x = L) = 2h(x = 0) = GL$. Since the heat flow is defined through unit cross-section, in order to obtain the heat taken out of the bar we have to multiply this by A. Thus $Q_{out} = GLA$. On the other hand, the total heat pumped in is $Q_{in} = GV = GLA$, hence these two are equal.

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† **Exercise 5.18.**   On one hand, the probability of having a collision exactly after time t is given by $P(t)dt$, while this can also be expressed as having no collisions for time t as

$$P(t)dt = \left(1 - \frac{dt}{\tau}\right)^{\frac{t}{dt}} \frac{dt}{\tau} . \tag{1}$$

The first bracket raised to the power t/dt is nothing but the probability of not having collisions for a number of t/dt time periods of duration dt. By denoting $x = \tau/dt$, we can re-cast the right hand side

$$\left(1 - \frac{dt}{\tau}\right)^{\frac{t}{dt}} \frac{dt}{\tau} = \left(1 - \frac{1}{x}\right)^{\frac{t}{\tau}x} \frac{dt}{\tau} = e^{-\frac{t}{\tau}} \frac{dt}{\tau} , \tag{2}$$

from which we conclude that

$$P(t) = \frac{e^{-\frac{t}{\tau}}}{\tau} , \tag{3}$$

which also means that the average time since the last collision is τ, and consequently, the mean collision time is 2τ.

On average, one electron loses $\dfrac{1}{2}mv^2 = \dfrac{2F^2\tau^2}{m}$ energy in 2τ, and in a volume of $A \cdot l$ we have $nA \cdot l$ electrons, where n is the electron density. This implies that the average dissipation per unit time is

$$P = nAl\frac{F^2\tau}{m} = nAl\frac{(eE)^2\tau}{m} = Al\frac{ne^2\tau}{m}\left(\frac{V}{l}\right)^2 = \frac{A\sigma}{l}V^2 = \frac{V^2}{R} \tag{4}$$

thus, the two terms with the ~~~ cancel each other. The $(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p})$ term can be expanded as

$$
\begin{aligned}
(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p}) &= (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3)(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3) \\[2mm]
&= \alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + \alpha_3^2 p_3^2 + \underbrace{(\alpha_1 \alpha_2 + \alpha_2 \alpha_1)}_{=0} p_1 p_2 \\[2mm]
&\quad + \underbrace{(\alpha_1 \alpha_3 + \alpha_3 \alpha_1)}_{=0} p_1 p_3 + \underbrace{(\alpha_2 \alpha_3 + \alpha_3 \alpha_2)}_{=0} p_2 p_3 \\[2mm]
&= \alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + \alpha_3^2 p_3^2 \ ,
\end{aligned}
\tag{3}
$$

where we used the anticommutation relation stipulated in Eq. (10.86). Then we are left with

$$
(E + \alpha_0 mc^2 + c\vec{\alpha} \cdot \vec{p})(E - \alpha_0 mc^2 - c\vec{\alpha} \cdot \vec{p}) = E^2 - (mc^2)^2 - |c\vec{p}|^2 \ ,
\tag{4}
$$

as stated.

Exercise 10.16.   Let the three momenta be $\vec{p}_\gamma, \vec{p}_e, \vec{p}_p$ for the photon, electron and positron, respectively, and likewise for the energies, E_γ, E_e, E_p. The three energies are linked to the momenta and the masses through the dispersion relations

$$
E_\gamma = \sqrt{(\vec{p}_\gamma c)^2}
\tag{1}
$$

$$
E_e = \sqrt{(m_e c^2)^2 + (\vec{p}_e c)^2}
\tag{2}
$$

$$
E_p = \sqrt{(m_e c^2)^2 + (\vec{p}_p c)^2} \ ,
\tag{3}
$$

where m_e is the mass of the electron and positron. The conservation of energy requires

$$
E_\gamma = E_e + E_p \ ,
\tag{4}
$$

or

$$
E_\gamma^2 = E_e^2 + E_p^2 + 2 E_e E_p \ ,
\tag{5}
$$

which, keeping the relation between momentum and energy in mind, can be expanded as

$$
\vec{p}_\gamma^2 = (m_e c)^2 + (m_e c)^2 + p_e^2 + p_p^2 + 2\sqrt{(m_e c)^2 + p_e^2}\sqrt{(m_e c)^2 + p_p^2} \ .
\tag{6}
$$

On the other hand, momentum is conserved in the process, i.e.,

$$
\vec{p}_\gamma = \vec{p}_e + \vec{p}_p \ ,
\tag{7}
$$

or

$$
p_\gamma^2 = p_e^2 + p_p^2 + 2\vec{p}_e \cdot \vec{p}_p \ .
\tag{8}
$$