Solutions Manual

to accompany

Probability, Random Variables and Stochastic Processes

Fourth Edition

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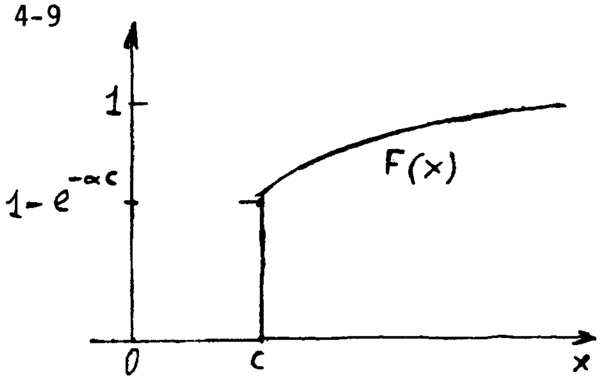
4-8 $\{(\underset{\sim}{x} - 10)^2 < 4\} = \{8 < \underset{\sim}{x} < 12\}$

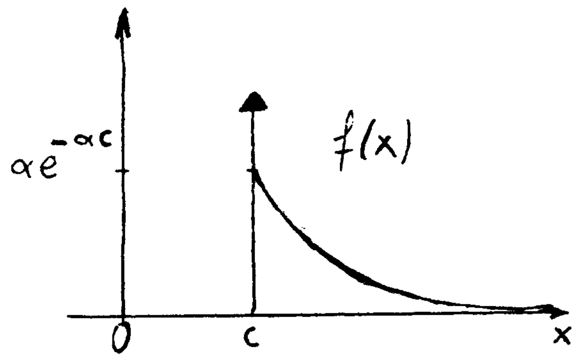
$P\{(\underset{\sim}{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$

$$f(x|(\underset{\sim}{x} - 10)^2 < 4) = \frac{f(x)}{P\{8 < \underset{\sim}{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}$$

for $8 < x < 12$ and zero otherwise

4-9





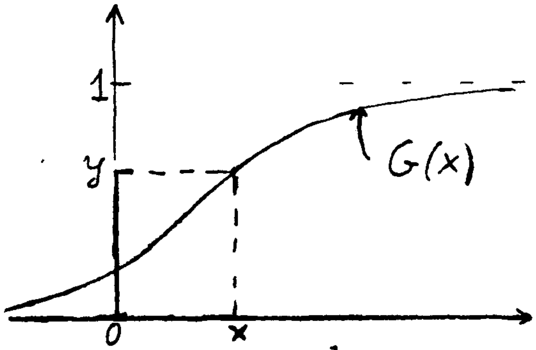
$F(x) = (1 - e^{-\alpha x})U(x-c)$ $\qquad$ $f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$

4-10 (a) $P\{1 \le x \le 2\} = G(\frac{2}{2}) - G(\frac{1}{2}) = 0.1499$

(b) $P\{1 \le \underset{\sim}{x} \le 2 | \underset{\sim}{x} \ge 1\} = \dfrac{G(1) - G(0.5)}{1 - G(0.5)} = \dfrac{0.1499}{0.3085} = 0.4857$

because $\{1 \le \underset{\sim}{x} \le 2, \underset{\sim}{x} \ge 1\} = \{1 \le \underset{\sim}{x} \le 2\}$

4-11



$x = G^{-1}(y) = H(y)$

If $\underset{\sim}{x}(t_1) \le x$

then

$t_1 \le y = G(x)$

Hence,

$P\{\underset{\sim}{x} \le x\} = P\{t_1 \le y\} = y = G(x)$

(d)

$$Z = X/Y$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{\tfrac{X}{Y} \le z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz,y)dy = \int_0^\infty y e^{y(z+1)}dy = \int_0^\infty y e^{(1+z)y} \\ &= \left[y\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right)\int_0^\infty e^{(1+z)y}dy \\ &= \left(\frac{1}{1+z}\right)\left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2}\,U(z) \end{aligned}$$

(e)

$$Z = \min\,(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\min\,(X,Y) \le z\} \\ &= 1 - P\{Z > z, Y > z\} \\ &= 1 - [1 - F_X(z)]\,[1 - F_Y(z)] \\ &= F_X(z) + F_Y(z) - F_X(z)\,F_Y(z) \end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z)f_Y(z) - f_X(z)F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z}\,U(z)$$

so that

$$F_X(z) = \int_0^z e^{-x}dx = (1 - e^{-z})\,U(z) = F_Y(z)$$

$$\begin{aligned} f_Z(z) &= [e^{-z} + e^{-z} - 2(1-e^{-z})\,e^{-z}]U(z) \\ &= 2e^{-z}\,[1 - 1 + e^{-z}]\,U(z) \\ &= 2e^{-2z}\,U(z) \sim \text{Exponential (2)}. \end{aligned}$$

(f)

$$Z = \max(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\ &= P\{X \le z\}\,P\{Y \le z\} = F_X(z)\,F_Y(z) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= F_X(z)\,f_Y(z) + f_X(z)\,F_Y(z) \\ &= e^{-z}\,(1-e^{-z}) + e^{-z}\,(1-e^{-z}) \\ &= 2e^{-z}\,(1-e^{-z})\,U(z) \end{aligned}$$

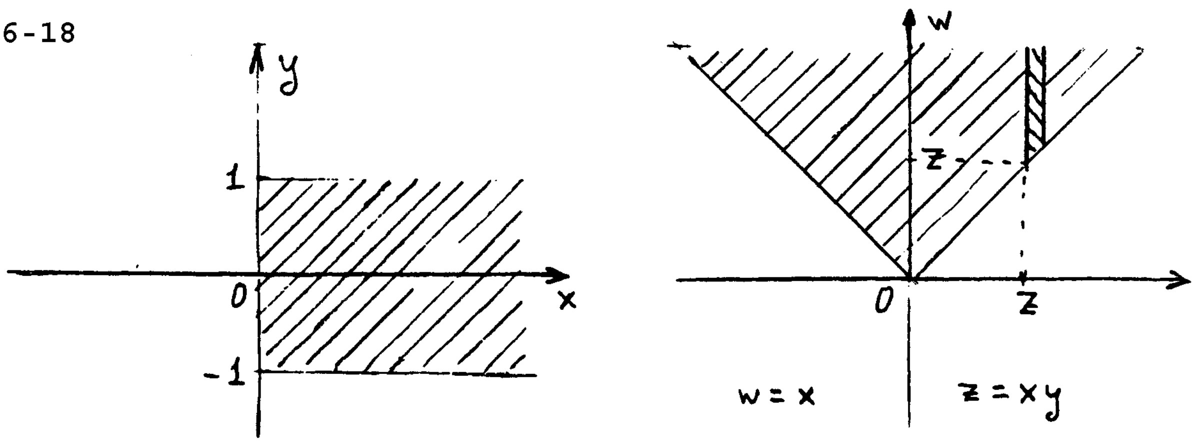
6-17 (a) If $\underset{\sim}{z} = 2\underset{\sim}{x} + 3\underset{\sim}{y}$ then $E\{\underset{\sim}{z}\} = 0$ $\quad \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

Hence, $\underset{\sim}{z}$ is $N(0;\sqrt{52})$

(b) If $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z \qquad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18



$$f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \qquad x = w \qquad y = z/w$$

The function $f_{zw}(z,w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi\alpha^2} \int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1 - z^2/w^2}}$$

$$= \frac{1}{\pi\alpha^2} \int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2}\, ds = \frac{1}{\alpha\sqrt{2\pi}} e^{-z^2/2\alpha^2}$$

6.45 (a) Let

$$Z = \min(X, Y), \qquad W = X - Y$$

$$\begin{aligned}
& P\{Z = k, W = m\} \\
& = P\{\min(X, Y) = k, X - Y = m\} \\
& = P\{(\min(X, Y) = k, X - Y = m) \bigcap (X \geq Y \bigcup X < Y)\} \\
& = P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\} \\
& = P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\}
\end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned}
P\{Z = k, W = m\} &= \begin{cases} P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0,\ m \geq 0 \\ P\{X = k, Y = k - m, X < Y\}, & k \geq 0,\ m < 0 \end{cases} \\
&= \begin{cases} pq^{k+m}\, pq^k, & k \geq 0,\ m \geq 0 \\ pq^k\, pq^{k-m}, & k \geq 0,\ m < 0 \end{cases}
\end{aligned}$$

$$P\{Z = k, W = m\} = p^2 q^{2k+|m|}, \qquad k = 0, 1, 2, \cdots, \quad m = 0, \pm 1, \pm 2, \cdots$$

Also

$$\begin{aligned}
P\{Z = k\} &= \sum_{m=-\infty}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) \\
&= p^2 q^{2k} \left(1 + \frac{2q}{p}\right) = p(1+q) q^{2k}, \qquad k = 0, 1, 2, \cdots
\end{aligned}$$

and

$$\begin{aligned}
P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{|m|} \sum_{k=0}^{\infty} q^{2k} \\
&= \frac{p}{1+q} q^{|m|}, \qquad m = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

7-5 (a) The joint density f(x,y) has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2+y^2+z^2})\,dz$$

depends only on $x^2 + y^2$. The same holds for f(x,z) and f(y,z). And since the RVs $\mathbf{x}$,$\mathbf{y}$, and $\mathbf{z}$ are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\mathbf{v}_x$,$\mathbf{v}_y$,$\mathbf{v}_z$ are $N(0; \sqrt{kT/m})$. With $\sigma^2 = kT/m$ and $n = 3$ it follows from (7-62)-(7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m^3}{\pi k^3 T^3}}\; v^2 e^{-mv^2/2kT}\, U(v)$$

$$E\{\mathbf{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \qquad E\{\mathbf{v}^{2n}\} = 1 \times 3 \cdots (2n+1)\left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\mathbf{y} = a\mathbf{x} + b$, $\mathbf{z} = c\mathbf{y} + d$, hence,

$$\mathbf{z} = A\mathbf{x} + B \qquad \eta_z = A\eta_x + B \qquad \sigma_z = A\sigma_x$$

$$E\{(\mathbf{z}-\eta_z)(\mathbf{x}-\eta_x\} = E\{A(\mathbf{x}-\eta_x)(\mathbf{x}-\eta_x)\} = A\sigma_x^2 = \sigma_x\sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming $\mathbf{x}_3$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E^2\{(\mathbf{x}[n+m+1] - \mathbf{x}[n+m])\mathbf{x}[n]\} \le E\{|\mathbf{x}[n+m+1] - \mathbf{x}[n+m]|^2\}E\{|\mathbf{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \le 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m.

9-51 We shall show that

$$2\,\frac{R^2[1]}{R[0]} - R[0] \le R[2] \le R[0] \qquad \text{(i)}$$

The covariance matrix of the RVs $\mathbf{x}[n]$, $\mathbf{x}[n+1]$, and $\mathbf{x}[n+2]$ is non-negative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \ge 0$$

This yields

$$R[0]R^2[2] - 2\,R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \le 0$$

The above is a quadratic in $R[2]$ with roots

$$R[0] \text{ and } -R[0] + 2\,R^2[1]/R[0]$$

Since it is nonpositive, $R[2]$ must be between the roots as in (i)

9-52 If $\mathbf{x}[n] = Ae^{jn\boldsymbol{\omega}T}$ then

$$R_x[m] = A^2E\{e^{j(m+n)\boldsymbol{\omega}T}e^{-jn\boldsymbol{\omega}T}\} = A^2\int_{-\sigma}^{\sigma} e^{jm\omega T}\, f(\omega)\,d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} S_x(\omega)e^{jm\omega T}d\omega$$

hence, $A^2 f(\omega) = S_x(\omega)/2\sigma$

14-11 From Prob. 10-10 it follows with $g(x) = x$ that $H(\underline{x},\underline{x}) = H(\underline{x})$. And since [see (14-103)] $H(\underline{x},\underline{x}) = H(\underline{x}|\underline{x}) + H(\underline{x})$ we conclude that $H(\underline{x}|\underline{x}) = 0$. From Prob. 14-3 it follows that

$$H(\underline{y},\underline{x}|\underline{x}) = H(A_y \cdot A_x | A_x) = H(A_x \cdot A_x) + H(A_y | A_x \cdot A_x)$$

$$= H(A_y|A_x) = H(\underline{y}|\underline{x})$$

because $A_x \cdot A_x = A_x$ and $H(A_x \cdot A_x) = H(\underline{x},\underline{x}) = 0$.

14-12 $E\{\underline{x}_n\} = 0 \qquad E\{\underline{x}_n^2\} = 5 \qquad E\{\underline{y}_n\} = 0$

$$E\{\underline{y}_n^2\} = \sum_{k=0}^{\infty} 2^{-2k} E\{\underline{x}_{n-k}^2\} = \frac{20}{3} \qquad E\{\underline{x}_n\underline{y}_n\} = E\{\underline{x}_n^2\} = 5$$

(a) From (14-95), (14-84), and (15-86) with $\mu_{11} = 5$, $\mu_{22} = 20/3$, and $\mu_{12} = 5$

$$H(\underline{x}) = \ln\sqrt{10\pi e} \qquad H(\underline{y}) = \ln\sqrt{40\pi e/3} \qquad H(\underline{x},\underline{y}) = \ln 10\pi e/\sqrt{3}$$

$$I(\underline{x},\underline{y}) = \ln 2$$

(b) The process $\underline{y}(t)$ is the output of the system

$$L(z) = \frac{1}{1 - 0.5\,z^{-1}} \qquad \ell_o = 1$$

with input $\underline{x}_n$. Since $\bar{H}(\underline{x}) = H(\underline{x})$ and [see (12A-1)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|L(e^{j\phi})|\,d\phi = \ln\ell_o = 0$$

(14-133) yields $\bar{H}(\underline{y}) = \bar{H}(\underline{x}) = H(\underline{x}) = \ln\sqrt{10\pi e}$.
