Solution Manual
for
Modern Electrodynamics

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A Note from the Author

This manual provides solutions to the end-of-chapter problems for the author's *Modern Electrodynamics*. The chance that all these solutions are correct is zero. Therefore, I will be pleased to hear from readers who discover errors. I will also be pleased to hear from readers who can provide a better solution to this or that problem than I was able to construct. I urge readers to suggest that this or that problem *should not* appear in a future edition of the book and (equally) to propose problems (and solutions) they believe *should* appear in a future edition.

At a fairly advanced stage in the writing of this book, I decided that a source should be cited for every end-of-chapter problem in the book. Unfortunately, I had by that time spent a decade accumulating problems from various places without always carefully noting the source. For that reason, I encourage readers to contact me if they recognize a problem of their own invention or if they can identify the (original) source of any particular problem in the manual. An interesting issue arises with problems I found on instructor or course websites which were taken down after the course they serviced had concluded. My solution has been to cite the source of these problems as a "public communication" between myself and the course instructor. This contrasts with problems cited as a true "private communication" between myself and an individual.

Chapter 1: Mathematical Preliminaries

1.1 Levi-Cività Practice I

(a) $\epsilon_{123} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1$. The cyclic property of the triple scalar product guarantees that $\epsilon_{231} = \epsilon_{312} = 1$ also. Similarly, $\epsilon_{132} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2) = -\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = -1$ with $\epsilon_{321} = \epsilon_{213} = -1$ also. Finally, $\epsilon_{122} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2) = 0$ and similarly whenever two indices are equal.

(b) Expand the determinant by minors to get

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{e}}_1(a_2 b_3 - a_3 b_2) - \hat{\mathbf{e}}_2(a_1 b_3 - a_3 b_1) + \hat{\mathbf{e}}_3(a_1 b_2 - a_2 b_1).$$

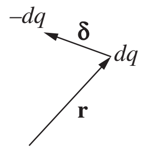
Using the Levi-Cività symbol to supply the signs, this is the same as the suggested identity because

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \epsilon_{123}\hat{\mathbf{e}}_1 a_2 b_3 + \epsilon_{132}\hat{\mathbf{e}}_1 a_3 b_2 \\ &\quad + \epsilon_{213}\hat{\mathbf{e}}_2 a_1 b_3 + \epsilon_{231}\hat{\mathbf{e}}_2 a_3 b_1 \\ &\quad + \epsilon_{312}\hat{\mathbf{e}}_3 a_1 b_2 + \epsilon_{321}\hat{\mathbf{e}}_3 a_2 b_1. \end{aligned}$$

(c) To get a non-zero contribution to the sum, the index i must be different from the unequal indices j and k, and also different from the unequal indices s and t. Therefore, the pair (i, j) and the pair (s, t) are the same pair of different indices. There are only two ways to do this. If $i = s$ and $j = t$, the ϵ terms are identical and their square is 1. This is the first term in the proposed identity. The other possibility introduces a transposition of two indices in one of the epsilon factors compared to the previous case. This generates an overall minus sign and thus the second term in the identity.

(d) The scalar of interest is $S = \hat{L}_m a_m \hat{L}_p b_p - \hat{L}_q b_q \hat{L}_s a_s$. Using the given commutation relation,

$$\begin{aligned} S &= a_m b_p \hat{L}_m \hat{L}_p - a_p b_m \hat{L}_m \hat{L}_p \\ &= a_m b_p \hat{L}_m \hat{L}_p - a_m b_p \hat{L}_p \hat{L}_m \\ &= a_m b_p [\hat{L}_m, \hat{L}_p] \\ &= i\hbar \epsilon_{mpi} \hat{L}_i a_m b_p \\ &= i\hbar \hat{L}_i \epsilon_{imp} a_m b_p \\ &= i\hbar \hat{\mathbf{L}} \cdot (\mathbf{a} \times \mathbf{b}). \end{aligned}$$



We will take the limit $\boldsymbol{\delta} \to 0$ presently so it is appropriate to perform a Taylor expansion to get

$$dW = \varphi(\mathbf{r})dq - [\varphi(\mathbf{r}) + \boldsymbol{\delta} \cdot \nabla\varphi(\mathbf{r})]dq = -\boldsymbol{\delta}q \cdot \nabla\varphi(\mathbf{r}).$$

From the figure, it is consistent to define $d\mathbf{p} = -\boldsymbol{\delta}dq$ in the limit when $dq \to \infty$ and $\boldsymbol{\delta} \to 0$ such that their product remains finite. Therefore, because $\mathbf{E} = -\nabla\varphi$, we get the desired result,

$$dW = -\mathbf{E}(\mathbf{r}) \cdot d\mathbf{p}.$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

4.7 Dipoles at the Vertices of Platonic Solids

The electric field of a point dipole is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}}{|\mathbf{r} - \mathbf{r}_0|^3} - \frac{4\pi}{3}\mathbf{p}\,\delta(\mathbf{r} - \mathbf{r}_0)\right],$$

where $\hat{\mathbf{n}} = (\mathbf{r} - \mathbf{r}_0)/|\mathbf{r} - \mathbf{r}_0|$. The delta function has no effect since we are interested in the field $\mathbf{E}(0)$ at the center of each polyhedron. Also, $\hat{\mathbf{n}} = \hat{\mathbf{r}}_0$ at this observation point.

(a) The positions $\mathbf{r}_0$ of the dipoles for the octahedron on the far left can be taken to be $\pm a\hat{\mathbf{x}}$, $\pm a\hat{\mathbf{y}}$, and $\pm a\hat{\mathbf{z}}$. Therefore, $r_0 = a$ and $\hat{\mathbf{n}}$ takes the values $\pm a\hat{\mathbf{x}}$, $\pm a\hat{\mathbf{y}}$, and $\pm a\hat{\mathbf{z}}$ when we sum over dipoles. Hence, the total field at the origin is

$$\begin{aligned}\mathbf{E}(0) &= \frac{1}{4\pi\epsilon}\frac{1}{a^3}\left[-6\mathbf{p} + 3\hat{\mathbf{x}}p_x + 3(-\hat{\mathbf{x}})(-p_x) + 3\hat{\mathbf{y}}p_y + 3(-\hat{\mathbf{y}})(-p_y) + 3\hat{\mathbf{z}}p_z + 3(-\hat{\mathbf{z}})(-p_z)\right] \\ &= 0.\end{aligned}$$

(b) The positions $\mathbf{r}_0$ of the dipoles for the tetrahedron in the middle are $a(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$, $a(-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}})$, $a(-\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}})$, and $a(\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}})$. Therefore, $r_0 = \sqrt{3}a$ and $\hat{\mathbf{n}}$ takes the values $(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})/\sqrt{3}$, $(-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}})/\sqrt{3}$, $(-\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}})/\sqrt{3}$, and $(\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}})\sqrt{3}$. Hence, the total field at the origin is

$$\begin{aligned}\mathbf{E}(0) &= \frac{1}{4\pi\epsilon_0}\frac{1}{3a^3}\left[-4\mathbf{p} + (\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})(p_x + p_y + p_z) + (-\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}})(-p_x - p_y + p_z)\right] \\ &+ \frac{1}{4\pi\epsilon_0}\frac{1}{3a^3}\left[(-\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}})(-p_x + p_y - p_z) + (\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}})(p_x - p_y - p_z)\right] \\ &= 0.\end{aligned}$$

(c) The eight dipoles at the corners of the cube are the superposition of two tetrahedra with dipoles at their corners rotated by 90° with respect to one another. From part (b), each tetrahedron contributes zero to the electric field at the center. Hence, $\mathbf{E}(0) = 0$ for this case also.

Therefore,

$$\mathbf{E}_{\rm out} - \mathbf{E}_{\rm in} = \frac{\sigma_{\rm P}}{\epsilon_0}\hat{\mathbf{n}} = \frac{\hat{\mathbf{n}}\cdot\mathbf{P}}{\epsilon_0}\hat{\mathbf{n}}.$$

Using $d\mathbf{S} = dS\hat{\mathbf{n}}$ and restoring the free charge gives the final result:

$$\mathbf{F} = \int d^3r\, [\rho_f(\mathbf{r}) + \mathbf{P}(\mathbf{r})\cdot\nabla]\mathbf{E}(\mathbf{r}) + \frac{1}{2\epsilon_0}\int d\mathbf{S}\, [\hat{\mathbf{n}}(\mathbf{r}_S)\cdot\mathbf{P}(\mathbf{r}_S)]^2\, .$$

6.25 Minimizing the Total Energy Functional

Using the hint, we seek a minimum of the functional

$$F[\mathbf{D}\,] = \frac{1}{2}\int_V d^3r\, \frac{|\mathbf{D}|^2}{\epsilon} - \int_V d^3r\, \varphi(\mathbf{r})(\nabla\cdot\mathbf{D} - \rho_f).$$

The factor of $\frac{1}{2}$ and the minus sign are inserted for convenience. Operationally, we compute $\delta F = F\,[\mathbf{D} + \delta\mathbf{D}] - F\,[\mathbf{D}]$ and look for the conditions that make $\delta F = 0$ to first order in $\delta\mathbf{D}$. This extremum is a minimum if $\delta F > 0$ to second order in $\delta\mathbf{D}$.

The first step is to integrate by parts to get

$$F[\mathbf{D}\,] = \frac{1}{2}\int_V d^3r\, \frac{|\mathbf{D}|^2}{\epsilon} + \int_V d^3r\, [\mathbf{D}\cdot\nabla\varphi + \rho_f\varphi] - \int_S d\mathbf{S}\cdot\mathbf{D}\,\varphi.$$

A direct calculation of δF to first order in $\delta\mathbf{D}$ gives

$$\delta F = \int_V d^3r\, \left[\frac{\mathbf{D}}{\epsilon} + \nabla\varphi\right]\cdot\delta\mathbf{D} - \int_S d\mathbf{S}\cdot\delta\mathbf{D}\,\varphi.$$

Finally, since the variation $\delta\mathbf{D}$ is arbitrary, δF vanishes if $\mathbf{D}(\mathbf{r}) = -\epsilon\nabla\varphi(\mathbf{r})$ and $\hat{\mathbf{n}}\cdot\delta\mathbf{D}|_S = 0$. The second of these is true if we specify the normal component of $\mathbf{D}$ on the boundary surface. The first implies that $\nabla\times\mathbf{D} = 0$. Together with the divergence constraint, this guarantees that $\mathbf{D}$ and $\mathbf{E} = -\nabla\varphi$ satisfy Maxwell's electrostatic equations. The second-order term in the variation of $F[\mathbf{D}]$ is $\frac{1}{2}\int d^3r\, |\delta\mathbf{D}|^2/\epsilon$. This is a positive quantity, so the extremum we have found does indeed correspond to a minimum of the total electrostatic energy.

Chapter 11: Magnetic Multipoles

11.1 Magnetic Dipole Moment Practice

We will find the current using $\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$. First,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 A_0}{4\pi}\left[\hat{\mathbf{r}}\frac{2\cos\theta}{r^2} + \hat{\boldsymbol{\theta}}\frac{\lambda \sin\theta}{r}\right]\exp(-\lambda r).$$

Therefore,

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \hat{\boldsymbol{\phi}} 4\pi A_0 \sin\theta \left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

The associated magnetic moment is

$$\mathbf{m} = \tfrac{1}{2}\int d^3 r\, \mathbf{r} \times \mathbf{j} = -\frac{A_0}{8\pi}\int d^3 r\, \hat{\boldsymbol{\theta}}\, r \sin\theta \left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

But $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$. This shows that only the $\hat{\mathbf{z}}$-component survives the integration. Hence,

$$\begin{aligned}
\mathbf{m} &= \hat{\mathbf{z}}\frac{A_0}{4}\int_0^\pi d\theta\, \sin^2\theta \int_0^\infty dr r^3 \exp(-\lambda r)\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\} \\
&= \hat{\mathbf{z}}\frac{\pi A_0}{8}\left\{2 - \lambda^2 \frac{d^2}{d\lambda^2}\right\}\int_0^\infty dr \exp(-\lambda r) \\
&= 0.
\end{aligned}$$

11.2 Origin Independence of Magnetic Multipole Moments

(a) If we shift the origin by a vector $\mathbf{d}$, the new magnetic moment is

$$\mathbf{m}' = \int d^3 r\, (\mathbf{r} - \mathbf{d}) \times \mathbf{j} = \mathbf{m} - \mathbf{d} \times \int d^3 r\, \mathbf{j} = \mathbf{m}.$$

The last equality above is true by conservation of charge. In the language of current loops,

$$\int d^3 r\, \mathbf{j} = I \oint d\mathbf{s} = 0.$$

(b) Similarly, $m_{ij}^{\prime(2)} = \dfrac{1}{3}\displaystyle\int d^3 r\, [(\mathbf{r} - \mathbf{d}) \times \mathbf{j}]_i (\mathbf{r} - \mathbf{d})_j$. Writing out the four terms gives

$$m_{ij}^{\prime(2)} = \frac{1}{3}\int d^3 r\, (\mathbf{r}\times\mathbf{j})_i r_j - \frac{1}{3}\int d^3 r\, (\mathbf{r}\times\mathbf{j})_i d_j - \frac{1}{3}\int d^3 r\, (\mathbf{d}\times\mathbf{j})_i r_j + \frac{1}{3}\int d^3 r\, (\mathbf{d}\times\mathbf{j})_i d_j.$$

$$\rho(\mathbf{r},t) = \sum_k q_k \delta(\mathbf{r}-\mathbf{r}_k).$$

Therefore,

$$\int d^3r \, \rho \mathbf{A}_\perp = \sum_k q_k \mathbf{A}_\perp(\mathbf{r}_k,t),$$

and we conclude that

$$\mathbf{P}_{\rm EM} = \sum_k q_k \mathbf{A}_\perp(\mathbf{r}_k,t) + \epsilon_0 \int d^3r \mathbf{E}_\perp \times \mathbf{B}.$$

Source: M.G.Calkin, *American Journal of Physics* **34**, 921 (1966).

15.21 Hidden Momentum in a Bar Magnet?

(a) For a permanent magnet, $\mathbf{B} = \mu_0(\mathbf{M}+\mathbf{H})$ and $\mathbf{H} = -\nabla\psi$, where ψ is the magnetic scalar potential. Therefore,

$$\mathbf{P}_{\rm EM} = \epsilon_0 \int d^3r \, \mathbf{E}\times\mathbf{B} = \frac{1}{c^2}\int d^3r \, \mathbf{E}\times(\mathbf{M}-\nabla\psi).$$

The $\nabla\psi$ term is zero because integration by parts produces a factor of $\nabla\times\mathbf{E}$ in the integrand. This is zero because the point charge is at rest. Therefore,

$$\mathbf{P}_{\rm EM} = \frac{1}{c^2}\int d^3r \, \mathbf{E}\times\mathbf{M}.$$

(b) The center-of-energy theorem surely demands $\mathbf{P}_{\rm tot} = 0$ for this situation. If so, some hidden momentum to cancel $\mathbf{P}_{\rm EM}$ is required. However, there are no "moving parts". The magnetic moment due to spin is a relativistic effect, but its origin is quantum-mechanical, rather than classical.

15.22 $\mathbf{L}_{\rm EM}$ for a Charge in a Two-Dimensional Magnetic Field

$$\mathbf{L}_{\rm EM} = \epsilon_0 \int d^3r \, \mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \epsilon_0 \int d^3r \, \left[\mathbf{E}(\mathbf{r}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{r}\cdot\mathbf{E})\right].$$

We have $\mathbf{B} = B(x,y)\hat{\mathbf{z}}$ and

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r^3}.$$

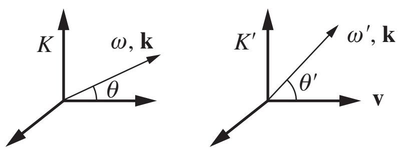
Therefore,

mode does not generate a longitudinal magnetic field. The TE/TM classification is a Lorentz invariant concept.

Source: M. Aalund and G. Johannsen, *Journal of Applied Physics* **42**, 2669 (1971).

22.20 Stellar Aberration

The geometry is



The transformation law for the four-vector $\vec{k}$ is

$$\mathbf{k}_\perp = \mathbf{k}'_\perp$$

$$\mathbf{k}_{||} = \gamma(\mathbf{k}'_{||} + \boldsymbol{\beta} k'_0)$$

$$k_4 = \gamma(k'_4 + \boldsymbol{\beta} \cdot \mathbf{k}'_{||}).$$

Therefore,

$$k_{||} = \gamma(k'_{||} + v\omega'/c^2) \qquad \text{and} \qquad \mathbf{k}_\perp = \mathbf{k}'_\perp.$$

It is most convenient to compute the inverse. Using $\omega' = ck'$,

$$\cot\theta = \frac{k_{||}}{k_\perp} = \frac{\gamma k'_{||} + \gamma v c k'/c^2}{k'_\perp} = \gamma\cot\theta' + \frac{\gamma v k'}{ck'\sin\theta'} = \gamma\left(\frac{\cos\theta'}{\sin\theta'} + \frac{\beta}{\sin\theta'}\right).$$

Therefore,

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}.$$

22.21 Reflection from a Rotating Mirror

The frequency ω and wave vector $\mathbf{k}$ of a monochromatic plane wave form a four-vector. Therefore, if the inertial frame S' moves with velocity $\mathbf{v}$ with respect to the (lab) frame S,

$$\omega' = \gamma(\omega - \mathbf{v}\cdot\mathbf{k})$$

$$\mathbf{k}'_{||} = \gamma(\mathbf{k}_{||} - \mathbf{v}\omega/c^2)$$

$$\mathbf{k}'_\perp = \mathbf{k}_\perp.$$

Our strategy is to (i) transform to the mirror frame; (ii) apply Snell's law of reflection; (iii) transform back to the lab frame. However, $\mathbf{v}$ lies in the plane of the mirror so $\mathbf{k}_\perp$ is the