

Solutions Manual to
MATHEMATICAL STATISTICS:
Asymptotic Minimax Theory

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$$= -\frac{1}{p_0(x-\theta)} \frac{\partial p_0(x-\theta)}{\partial x} = -\frac{p_0'(x-\theta)}{p_0(x-\theta)}.$$

Thus we can write

$$I_n(\theta) = n \mathbb{E}_\theta \left[\left(-\frac{p_0'(X-\theta)}{p_0(X-\theta)} \right)^2 \right] = n \int_{\mathbb{R}} \frac{(p_0'(y))^2}{p_0(y)} dy,$$

which is a constant independent of θ .

EXERCISE 1.8 Using the expression for the Fisher information derived in the previous exercise, we write

$$\begin{aligned} I_n(\theta) &= n \int_{\mathbb{R}} \frac{(p_0'(y))^2}{p_0(y)} dy = n \int_{-\pi/2}^{\pi/2} \frac{(-C\alpha \cos^{\alpha-1} y \sin y)^2}{C \cos^\alpha y} dy \\ &= n C \alpha^2 \int_{-\pi/2}^{\pi/2} \sin^2 y \cos^{\alpha-2} y dy = n C \alpha^2 \int_{-\pi/2}^{\pi/2} (1 - \cos^2 y) \cos^{\alpha-2} y dy \\ &= n C \alpha^2 \int_{-\pi/2}^{\pi/2} (\cos^{\alpha-2} y - \cos^\alpha y) dy. \end{aligned}$$

Here the first term is integrable if $\alpha - 2 > -1$ (equivalently, $\alpha > 1$), while the second one is integrable if $\alpha > -1$. Therefore, the Fisher information exists when $\alpha > 1$.

and, thus, for all sufficiently large n , the above integral admits the upper bound

$$\begin{aligned} & n^{1/2} \int_{-n^{1/4}}^{n^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-(u-\theta n^{1/2})^2/2} du \\ & \leq n^{1/2} \int_{-n^{1/4}}^{n^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2+\theta n^{3/4}-\theta^2 n/2} du \\ & \leq e^{-\theta^2 n/4} \int_{-n^{1/4}}^{n^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Further, we use the Cauchy-Schwarz inequality to write

$$\begin{aligned} & \mathbb{E}_\theta \left[n (\hat{\theta}_n - \theta)^2 \right] = \mathbb{E}_\theta \left[n (\hat{\theta}_n - \bar{X}_n + \bar{X}_n - \theta)^2 \right] \\ & = \mathbb{E}_\theta \left[n (\hat{\theta}_n - \bar{X}_n)^2 \right] + 2 \mathbb{E}_\theta \left[n (\hat{\theta}_n - \bar{X}_n)(\bar{X}_n - \theta) \right] + \mathbb{E}_\theta \left[n (\bar{X}_n - \theta)^2 \right] \\ & \leq \underbrace{\mathbb{E}_\theta \left[n (\hat{\theta}_n - \bar{X}_n)^2 \right]}_{\rightarrow 0} + 2 \underbrace{\left\{ \mathbb{E}_\theta \left[n (\hat{\theta}_n - \bar{X}_n)^2 \right] \right\}^{1/2}}_{\rightarrow 0} \times \\ & \quad \times \underbrace{\left\{ \mathbb{E}_\theta \left[n (\bar{X}_n - \theta)^2 \right] \right\}^{1/2}}_{=1} + \underbrace{\mathbb{E}_\theta \left[n (\bar{X}_n - \theta)^2 \right]}_{=1} \rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consider now the case $\theta = 0$. We will verify that

$$\lim_{n \rightarrow \infty} \mathbb{E}_\theta \left[n \hat{\theta}_n^2 \right] = 0.$$

We have

$$\begin{aligned} & \mathbb{E}_\theta \left[n \hat{\theta}_n^2 \right] = \mathbb{E}_\theta \left[n \bar{X}_n^2 \mathbb{I}(|\bar{X}_n| \geq n^{-1/4}) \right] \\ & = \mathbb{E}_\theta \left[(\sqrt{n} \bar{X}_n)^2 \mathbb{I}(|\sqrt{n} \bar{X}_n| \geq n^{1/4}) \right] = 2 \int_{n^{1/4}}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-z^2/2} dz \\ & \leq 2 \int_{n^{1/4}}^{\infty} e^{-z} dz = 2 e^{-n^{1/4}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

EXERCISE 3.16 The following lower bound holds:

$$\begin{aligned} & \sup_{\theta \in \mathbb{R}} \mathbb{E}_\theta \left[I_n(\theta) (\hat{\theta}_n - \theta)^2 \right] \geq n I_* \max_{\theta \in \{\theta_0, \theta_1\}} \mathbb{E}_\theta \left[(\hat{\theta}_n - \theta)^2 \right] \\ & \geq \frac{n I_*}{2} \left\{ \mathbb{E}_{\theta_0} \left[(\hat{\theta}_n - \theta_0)^2 \right] + \mathbb{E}_{\theta_1} \left[(\hat{\theta}_n - \theta_1)^2 \right] \right\} \\ & = \frac{n I_*}{2} \mathbb{E}_{\theta_0} \left[(\hat{\theta}_n - \theta_0)^2 + (\hat{\theta}_n - \theta_1)^2 \exp \left\{ \Delta L_n(\theta_0, \theta_1) \right\} \right] \quad (\text{by (3.8)}) \end{aligned}$$

(ii) From Exercise 4.23, θ_n^* is unbiased and its variance is equal to $1/n^2$. Hence,

$$\mathbb{E}_{\theta_0} \left[(n(\theta_n^* - \theta_0))^2 \right] = n^2 \text{Var}_{\theta_0} [\theta_n^*] = \frac{n^2}{n^2} = 1.$$

EXERCISE 4.28 Consider the case $y \leq 0$. Then

$$\begin{aligned} \lambda_0 \min_{y \leq 0} \int_0^\infty |u - y| e^{-\lambda_0 u} du &= \lambda_0 \min_{y \leq 0} \int_0^\infty (u - y) e^{-\lambda_0 u} du \\ &= \min_{y \leq 0} \left(\frac{1}{\lambda_0} - y \right) = \frac{1}{\lambda_0}, \text{ attained at } y = 0. \end{aligned}$$

In the case $y \geq 0$,

$$\begin{aligned} &\lambda_0 \min_{y \geq 0} \int_0^\infty |u - y| e^{-\lambda_0 u} du \\ &= \lambda_0 \min_{y \geq 0} \left(\int_y^\infty (u - y) e^{-\lambda_0 u} du + \int_0^y (y - u) e^{-\lambda_0 u} du \right) \\ &= \min_{y \geq 0} \left(\frac{2e^{-\lambda_0 y} - 1}{\lambda_0} + y \right) = \frac{\ln 2}{\lambda_0}, \end{aligned}$$

attained at $y = \ln 2 / \lambda_0$.

Thus,

$$\lambda_0 \min_{y \in \mathbb{R}} \int_0^\infty |u - y| e^{-\lambda_0 u} du = \min \left(\frac{\ln 2}{\lambda_0}, \frac{1}{\lambda_0} \right) = \frac{\ln 2}{\lambda_0}.$$

EXERCISE 4.29 (i) For a normalizing constant C , we write by definition

$$\begin{aligned} f_b(\theta | X_1, \dots, X_n) &= C f(X_1, \theta) \dots f(X_n, \theta) \pi_b(\theta) \\ &= C \exp \left\{ - \sum_{i=1}^n (X_i - \theta) \right\} \mathbb{I}(X_1 \geq \theta) \dots \mathbb{I}(X_n \geq \theta) \frac{1}{b} \mathbb{I}(0 \leq \theta \leq b) \\ &= C_1 e^{n\theta} \mathbb{I}(X_{(1)} \geq \theta) \mathbb{I}(0 \leq \theta \leq b) = C_1 e^{n\theta} \mathbb{I}(0 \leq \theta \leq Y) \end{aligned}$$

where

$$C_1 = \left(\int_0^Y e^{n\theta} d\theta \right)^{-1} = \frac{n}{\exp\{nY\} - 1}, \quad Y = \min(X_{(1)}, b).$$

Chapter 9

EXERCISE 9.63 If h_n does not vanish as $n \rightarrow \infty$, the bias of the local polynomial estimator stays finite. If nh_n is finite, the number of observations N within the interval $[x - h_n, x + h_n]$ stays finite, and can be even zero. Then the system of normal equations (9.2) either does not have a solution or the variance of the estimates does not decrease as n grows.

EXERCISE 9.64 Using Proposition 9.4 and the Taylor expansion (8.14), we obtain

$$\begin{aligned}\hat{f}_n(0) &= \sum_{m=0}^{\beta-1} (-1)^m \hat{\theta}_m = \left(\sum_{m=0}^{\beta-1} (-1)^m \frac{f^{(m)}(0)}{m!} h_n^m + \rho(0, h_n) \right) - \rho(0, h_n) + \\ &+ \sum_{m=0}^{\beta-1} (-1)^m (b_m + \mathcal{N}_m) = f(0) - \rho(0, h_n) + \sum_{m=0}^{\beta-1} (-1)^m b_m + \sum_{m=0}^{\beta-1} (-1)^m \mathcal{N}_m.\end{aligned}$$

Hence the absolute conditional bias of $\hat{f}_n(0)$ for a given design \mathcal{X} admits the upper bound

$$\left| \mathbb{E}_f[\hat{f}_n(0) - f(0)] \right| \leq |\rho(0, h_n)| + \sum_{m=0}^{\beta-1} |b_m| \leq \frac{Lh_n^\beta}{(\beta-1)!} + \beta C_b h_n^\beta = O(h_n^\beta).$$

Note that the random variables \mathcal{N}_m can be correlated. That is why the conditional variance of $\hat{f}_n(0)$, given a design \mathcal{X} , may not be computed explicitly but only estimated from above by

$$\begin{aligned}\mathbb{V}ar_f[\hat{f}_n(0) | \mathcal{X}] &= \mathbb{V}ar_f\left[\sum_{m=0}^{\beta-1} (-1)^m \mathcal{N}_m \middle| \mathcal{X}\right] \\ &\leq \beta \sum_{m=0}^{\beta-1} \mathbb{V}ar_f[\mathcal{N}_m | \mathcal{X}] \leq \beta C_v / N = O(1/N).\end{aligned}$$

EXERCISE 9.65 Applying Proposition 9.4, we find that the bias of $m! \hat{\theta}_m / (h_n^*)^m$ has the magnitude $O((h_n^*)^{\beta-m})$, while the random term $m! \mathcal{N}_m / (h_n^*)^m$ has the variance $O((h_n^*)^{-2m} (n h_n^*)^{-1})$. These formulas guarantee the optimality of $h_n^* = n^{-1/(2\beta+1)}$. Indeed, for any m ,

$$(h_n^*)^{2(\beta-m)} = (h_n^*)^{-2m} (n h_n^*)^{-1}.$$

So, the rate $(h_n^*)^{2(\beta-m)} = n^{-2(\beta-m)/(2\beta+1)}$ follows.

Chapter 12

EXERCISE 12.82 We have n design points in Q bins. That is why, for any design, there exist at least $Q/2$ bins with at most $2n/Q$ design points. Indeed, otherwise we would have strictly more than $(Q/2)(2n/Q) = n$ points. Denote the set of the indices of these bins by \mathcal{M} . By definition, $|\mathcal{M}| \geq Q/2$. In each such bin B_q , the respective variance is bounded by

$$\begin{aligned} \sigma_{q,n}^2 &= \sum_{x_i \in B_q} f_q^2(x_i) \leq \sum_{x_i \in B_q} (h_n^*)^{2\beta} \varphi^2\left(\frac{x_i - c_q}{h_n^*}\right) \\ &\leq \|\varphi\|_\infty^2 (h_n^*)^{2\beta} (2n/Q) = 4n \|\varphi\|_\infty^2 (h_n^*)^{2\beta+1} = 4 \|\varphi\|_\infty^2 \ln n \end{aligned}$$

which can be made less than $2\alpha \ln Q$ if we choose $\|\varphi\|_\infty$ sufficiently small.

EXERCISE 12.83 Select the test function defined by (12.3). Substitute M in the proof of Lemma 12.11 by Q , to obtain

$$\begin{aligned} \sup_{f \in \Theta(\beta)} \mathbb{E}_f \left[\psi_n^{-1} \|\hat{f}_n - f\|_\infty \right] &\geq d_0 \psi_n^{-1} \max_{1 \leq q \leq Q} \mathbb{E}_{f_q} \left[\mathbb{E}_{f_q} [\mathbb{I}(\mathcal{D}_q) \mid \mathcal{X}] \right] \\ &\geq d_0 \psi_n^{-1} \mathbb{E}^{(\mathcal{X})} \left[\frac{1}{2} \mathbb{P}_0(\mathcal{D}_0 \mid \mathcal{X}) + \frac{1}{2Q} \sum_{q=1}^Q \mathbb{P}_q(\mathcal{D}_q \mid \mathcal{X}) \right] \end{aligned}$$

where $\mathbb{E}^{(\mathcal{X})}[\cdot]$ denotes the expectation taken over the distribution of the random design.

Note that $d_0 \psi_n^{-1} = (1/2) \|\varphi\|_\infty$. Due to (12.22), with probability 1, for any random design \mathcal{X} , there exists a set $\mathcal{M}(\mathcal{X})$ such that

$$\frac{1}{2} \mathbb{P}_0(\mathcal{D}_0 \mid \mathcal{X}) + \frac{1}{2Q} \sum_{q=1}^Q \mathbb{P}_q(\mathcal{D}_q \mid \mathcal{X}) \geq \frac{|\mathcal{M}|}{4Q} \geq \frac{Q/2}{4Q} = \frac{1}{8}.$$

Combining these bounds, we get that

$$\sup_{f \in \Theta(\beta)} \mathbb{E}_f \left[\psi_n^{-1} \|\hat{f}_n - f\|_\infty \right] \geq (1/16) \|\varphi\|_\infty.$$

EXERCISE 12.84 The log-likelihood function is equal to

$$-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - f(x_i, \omega'))^2 + \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - f(x_i, \omega''))^2$$