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**Exercises and Solutions
Manual for**

Integration and Probability

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IV. Hilbert Space Methods and Limit Theorems in Probability Theory 95

- (1) Use Problem III-2 to show that the distribution of Y is independent of ν .
 (2) From now on, assume that the $\{X_j\}_{j=0}^n$ are independent, with the same distribution μ and with Fourier transform $\exp(-\frac{t^2}{2})$. Show, using Problem III-1, that μ must be invariant under every orthogonal matrix.
 (3) If $a \in \mathbf{R}$, compute the integral

$$I(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left[-\frac{1}{2} \left(x^2 + \frac{a^2}{x^2} \right) \right] dx$$

by using the following fact from Problem II-12(1):

$$\int_{-\infty}^{+\infty} f \left(x - \frac{|a|}{x} \right) dx = \int_{-\infty}^{+\infty} f(y) dy \quad \text{for every } f \text{ integrable on } \mathbf{R}.$$

- (4) By first conditioning with respect to X_0 (see Problem IV-34), compute the Fourier transform of the distribution of Y .
 (5) Using Problem IV-11, find the distribution of $\|Y\|^2$. Derive the density of Y from this, by observing that the distribution of Y is invariant under every orthogonal matrix in O_d and using Problem III-3.

SOLUTION. (1) By Problem III-2(4), if $U = (U_0, \dots, U_d)$ is a random variable concentrated on the unit sphere S_d of \mathbf{R}^{d+1} , independent of $\|X\|$ and with rotation-invariant distribution σ , then X and $U\|X\|$ have the same distribution. Hence Y and $(\frac{U_1}{U_0}, \frac{U_2}{U_0}, \dots, \frac{U_d}{U_0})$ have the same distribution, which proves the result.

(2) $\mathbf{E}[\exp(i \sum_{j=0}^d t_j X_j)] = \exp(-\frac{1}{2} \|t\|^2)$. Problem III-1 gives the result.

(3) Set $f(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})$. Then

$$f \left(x - \frac{|a|}{x} \right) = e^{+|a|} \exp \left(-\frac{1}{2} \left(x^2 + \frac{a^2}{x^2} \right) \right).$$

Hence $I(a) = e^{-|a|} I(0) = e^{-|a|}$.

$$(4) \quad \mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \right] = \mathbf{E} \left[\mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \mid X_0 \right] \right].$$

But, by (2),

$$\mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \mid X_0 \right] = \exp \left(-\frac{\|t\|^2}{2X_0^2} \right).$$

Using (3), the Fourier transform of the distribution of Y is thus $\exp(-\|t\|)$.

sub- σ -algebra \mathcal{C} of \mathcal{A} . Prove that if \mathcal{D} is the σ -algebra generated by $\mathcal{B} \cup \mathcal{C}$, then

$$\mathbf{E}[X|\mathcal{D}] = \mathbf{E}[X|\mathcal{B}].$$

METHOD. Prove the assertion first for square integrable X .

SOLUTION. If $X \in L^2(\mathcal{A})$, we must show that $Y = X - \mathbf{E}(X|B)$ is orthogonal not only to the subspace $L^2(\mathcal{B})$ of $L^2(\mathcal{A})$ but also to $L^2(\mathcal{D})$. For this, it suffices to show that Y is orthogonal to a dense subspace of $L^2(\mathcal{D})$. Since \mathcal{D} is generated by $\{B \cap C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$, a dense subspace of $L^2(\mathcal{D})$ is clearly formed by the set of those $Z \in \mathcal{D}$ for which there exist (i) a \mathcal{B} -measurable partition (B_1, \dots, B_n) of Ω , (ii) a \mathcal{C} -measurable partition (C_1, \dots, C_m) of Ω , and (iii) $(a_{ij})_{i=1}^n_{j=1}^m$ such that $Z = \sum_{i,j} a_{ij} \mathbf{1}_{B_i} \mathbf{1}_{C_j}$. It remains to show that $\mathbf{E}(YZ) = 0$. But

$$\begin{aligned} \mathbf{E}[YZ] &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B)) \mathbf{1}_{B_i} \mathbf{1}_{C_j}] \\ &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B)) \mathbf{1}_{B_i}] P[C_j], \end{aligned}$$

since Y is independent of \mathcal{C} . By definition, $\mathbf{E}(Y \mathbf{1}_{B_i}) = 0$. Hence $\mathbf{E}(YZ) = 0$.

Problem IV-36. If X and Y are integrable random variables such that $\mathbf{E}[X|Y] = Y$ and $\mathbf{E}[Y|X] = X$, show that $X = Y$ a.s.

METHOD. Show that, for fixed x ,

$$(i) \quad 0 \leq \int_{\{Y \leq x \leq X\}} (X - Y) dP = \int_{\{x < X \text{ and } x < Y\}} (Y - X) dP,$$

and conclude by symmetry that both sides of the equation are zero. Then use Problem I-13.

SOLUTION. Since $\mathbf{E}[X|X] = X$, we can write $\mathbf{E}[X - Y|X] = 0$; thus, for every Borel set A of \mathbf{R} ,

$$(ii) \quad \int_{X \in A} (X - Y) dP = 0.$$

Setting $A = \{X > x\} = \{Y \leq x < X\} \cup \{x < X \text{ and } x < Y\}$, (ii) implies (i). The positivity of the left-hand side is clear. Interchanging the roles of X and Y , the same reasoning gives

$$0 \leq \int_{\{x < X \text{ and } x < Y\}} (X - Y) dP.$$

This is the desired formula, since $C_n^k + C_n^{k-1} = C_{n+1}^k$.

(3) $(\exp(td))(\varphi)(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi^{(n)}(x) = \varphi(t+x)$ by Taylor's formula. It follows from (2) that

$$\begin{aligned} (\exp t(d+\rho))(\varphi)(x) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (d+\rho)^n \varphi(x) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{t^k}{k!} F_k(\rho) \frac{t^{n-k}}{(n-k)!} \varphi^{(n-k)}(x). \end{aligned}$$

But, by the definition of the F_k and formula V-1.4.2,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} F_k(x) = \exp \left[\frac{t^2}{2} + tx \right].$$

Thus, using the formula for the Cauchy product of two power series,

$$\begin{aligned} (\exp t(d+\rho))(\varphi)(x) &= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} F_n(\rho) \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi^{(n)}(x) \right) \\ &= \left(\exp \left(\frac{t^2}{2} + t\rho \right) \tau_t \right) (\varphi)(x). \end{aligned}$$

In particular, if $\lambda = 1$ then $\rho(\varphi) = x\varphi(x)$ and

$$(\exp t(d+x))(\varphi)(x) = \exp \left(\frac{t^2}{2} + tx \right) \varphi(t+x).$$

REMARK. The result of (2) is due to Viskov¹; that of (3) is due to Ville².

Problem V-4. Let X and Y be independent random variables with the same distribution $\nu_1(dx) = \exp(-\frac{x^2}{2}) \frac{dx}{\sqrt{2\pi}}$. Let $g : \mathbf{R} \rightarrow [0, +\infty)$ be a measurable function and let $Z = X + Y\sqrt{g(X)}$. Assume that Z has a normal distribution. Cantelli conjectured in 1917 that g is then constant almost everywhere; this is still unproved in 1994.

(1) Let $g_0 = \mathbf{E}(g(X))$. For all real t , compute $\mathbf{E}(\exp tZ)$ as a function of g_0 . Prove that $\exp(\alpha g) \in L^2(\nu_1)$ for all $\alpha > 0$.

METHOD. Use the Schwarz inequality.

(2) Let $\{g_n\}_{n=0}^{\infty}$ be the sequence of real numbers such that $g(x) = \sum_{n=0}^{\infty} g_n \frac{H_n(x)}{n!}$ in the $L^2(\nu_1)$ sense. By considering $\mathbf{E}(Z^3)$ and $\mathbf{E}(Z^4)$, show that $g_1 = 0$ and $-2g_2 = \sum_{n=2}^{\infty} \frac{g_n^2}{n!}$.

¹O. Viskov, *Theory of Probability and Its Applications*, Vol. 30, n. 1 (1984), 141–143.

²J. Ville, *Comptes Rendus Acad. des Sc.* 221 (1945), 529–539.