

Gérard Letac

**Exercises and Solutions
Manual for**

Integration and Probability

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Contents

I	Measurable Spaces and Integrable Functions	1
1	σ -algebras and partitions	1
2	r -families	3
3	Monotone classes and independence	3
4	Banach limits	4
5	A strange probability measure	5
6	Integration and distribution functions	6
7	Evaluating $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	8
8	Monotone convergence	9
9	Vector integration	10
10	Convergence in measure and composition of functions . .	11
11	Principle of separation of variables	12
12	The Cauchy-Schwarz inequality	13
13	Test that $X \geq Y$ almost everywhere	14
14	Image of a measure	14
15	Primitives of square integrable functions	15
II	Borel Measures and Radon Measures	17
1	Positive measures on an open interval	17
2	Distribution functions	20
3	Convexity and growth	21
4	Convexity and measure	22

vi Contents

5	Integral representation of positive convex functions on $(0, \infty)$	23
6	Integral representations of Askey functions	24
7	Gauss's inequality	26
8	Integral of a decreasing function	27
9	Second mean value theorem for integrals	27
10	Variance of a distribution on $[0, 1]$	28
11	Variance of the distribution of a convex function on $[0, 1]$	28
12	Rational functions which preserve Lebesgue measure	29
13	A measure on the half-plane	33
14	Weak convergence and moments	34
15	Improper integrals and Lebesgue measure	35
16	$\int_0^\infty \frac{\sin x}{x} dx, \int_0^\infty (\cos ax - \cos bx) \frac{dx}{x}, \int_0^\infty (\cos ax - \cos bx) \frac{dx}{x^2}$	36
17	Comparisons between different L^p spaces	36
18	Differentiation under the integral sign	37
19	Laplace transform of a measure on $[0, +\infty)$	40
20	Comparison of vague, weak, and narrow convergence	41
21	Weak compactness of measures	41
22	Vague convergence and limit of $\mu_n(0)$	41
23	Vague convergence and restriction to a closed set	43
24	Change of variables in an integral	45
25	Image of a measure and the Jacobian	45
III	Fourier Analysis	49
1	Characterizations of radial measures	49
2	Radial measures and independence	50
3	Area of the sphere	51
4	Fourier transform of the Poisson kernel of \mathbf{R}_+^{n+1}	52
5	Askey-Polya functions	53
6	Symmetric convex sets in the plane and measures on $[0, \pi)$	55
7	T. Ferguson's theorem	59
8	A counterexample of Herz	59
9	Riesz kernels	61
10	Measures on the circle and holomorphic functions	62
11	Harmonic polynomials and the Fourier transform	63
12	Bernstein's inequality	64
13	Cauchy's functional equation	66
14	Poisson's formula	66
15	A list of Fourier-Plancherel transforms	68
16	Fourier-Plancherel transform of a rational function	69
17	Computing some Fourier-Plancherel transforms	70
18	Expressing the Fourier-Plancherel transform as a limit	71
19	An identity for the Fourier-Plancherel transform	71
20	The Hilbert transform on $L^2(\mathbf{R})$	71

21	Action of $L^1(\mathbf{R})$ on $L^2(\mathbf{R})$	72
22	Another expression for the Hilbert transform	73
23	A table of properties of the Hilbert transform	73
24	Computing some Hilbert transforms	74
25	The Hilbert transform and distributions	74
26	Sobolev spaces on \mathbf{R}	75
27	H. Weyl's inequality	76

**IV Hilbert Space Methods and Limit Theorems
 in Probability Theory**

1	Fancy dice	79
2	The geometric distribution	80
3	The binomial and Poisson distributions	80
4	Construction of given distributions	83
5	Von Neumann's method	84
6	The laws of large numbers	85
7	Etemadi's method	86
8	A lemma on the random walks S_n	89
9	$\alpha(s) = \lim_{n \rightarrow \infty} (P[S_n \geq s \cdot n])^{1/n}$ exists	89
10	Evaluating $\alpha(s)$ in some concrete cases	90
11	Algebra of the gamma and beta distributions	92
12	The gamma distribution and the normal distribution	93
13	The Cauchy distribution and the normal distribution	94
14	A probabilistic proof of Stirling's formula	96
15	Maxwell's theorem	97
16	If X_1 and X_2 are independent, then $\frac{(X_1, X_2)}{(X_1^2 + X_2^2)^{1/2}}$ is uniform	98
17	Isotropy of pairs and triplets of independent variables	100
18	The only invertible distributions are concentrated at a point	102
19	Isotropic multiples of normal distributions	103
20	Poincaré's lemma	104
21	Schoenberg's theorem	104
22	A property of radial distributions	106
23	Brownian motion hits a hyperplane in a Cauchy distribution	107
24	Pittinger's inequality	109
25	Cylindrical probabilities	112
26	Minlos's lemma	113
27	Condition that a cylindrical probability be a probability measure	115
28	Lindeberg's theorem	116
29	H. Chernoff's inequality	117
30	Gebelein's inequality	120
31	Fourier transform of the Hermite polynomials	121

viii Contents

32	Another definition of conditional expectation	122
33	Monotone continuity of conditional expectations	123
34	Concrete computation of conditional expectations	123
35	Conditional expectations and independence	124
36	$\mathbf{E}(X Y) = Y$ and $\mathbf{E}(Y X) = X$	125
37	Warnings about conditional expectations	126
38	Conditional expectations in the absolutely continuous case and the Gaussian case	126
39	Examples of martingales	127
40	A reversed martingale	128
41	A probabilistic approximation of an arithmetic conjecture	129
42	A criterion for uniform integrability	129
43	The Galton-Watson process and martingales	130

V	Gaussian Sobolev Spaces and Stochastic Calculus of Variations	133
1	d and δ cannot both be continuous	133
2	Growth of the Hermite polynomials	134
3	Viskov's lemma	134
4	Cantelli's conjecture	136
5	Lancaster probabilities in \mathbf{R}^2	138
6	Sarmanov's theorem	139

IV. Hilbert Space Methods and Limit Theorems in Probability Theory 95

- (1) Use Problem III-2 to show that the distribution of Y is independent of ν .
 (2) From now on, assume that the $\{X_j\}_{j=0}^n$ are independent, with the same distribution μ and with Fourier transform $\exp(-\frac{t^2}{2})$. Show, using Problem III-1, that μ must be invariant under every orthogonal matrix.
 (3) If $a \in \mathbf{R}$, compute the integral

$$I(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left[-\frac{1}{2} \left(x^2 + \frac{a^2}{x^2} \right) \right] dx$$

by using the following fact from Problem II-12(1):

$$\int_{-\infty}^{+\infty} f \left(x - \frac{|a|}{x} \right) dx = \int_{-\infty}^{+\infty} f(y) dy \quad \text{for every } f \text{ integrable on } \mathbf{R}.$$

- (4) By first conditioning with respect to X_0 (see Problem IV-34), compute the Fourier transform of the distribution of Y .
 (5) Using Problem IV-11, find the distribution of $\|Y\|^2$. Derive the density of Y from this, by observing that the distribution of Y is invariant under every orthogonal matrix in O_d and using Problem III-3.

SOLUTION. (1) By Problem III-2(4), if $U = (U_0, \dots, U_d)$ is a random variable concentrated on the unit sphere S_d of \mathbf{R}^{d+1} , independent of $\|X\|$ and with rotation-invariant distribution σ , then X and $U\|X\|$ have the same distribution. Hence Y and $(\frac{U_1}{U_0}, \frac{U_2}{U_0}, \dots, \frac{U_d}{U_0})$ have the same distribution, which proves the result.

(2) $\mathbf{E}[\exp(i \sum_{j=0}^d t_j X_j)] = \exp(-\frac{1}{2} \|t\|^2)$. Problem III-1 gives the result.

(3) Set $f(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})$. Then

$$f \left(x - \frac{|a|}{x} \right) = e^{+|a|} \exp \left(-\frac{1}{2} \left(x^2 + \frac{a^2}{x^2} \right) \right).$$

Hence $I(a) = e^{-|a|} I(0) = e^{-|a|}$.

$$(4) \quad \mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \right] = \mathbf{E} \left[\mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \mid X_0 \right] \right].$$

But, by (2),

$$\mathbf{E} \left[\exp \left(i \sum_{j=1}^d t_j \frac{X_j}{X_0} \right) \mid X_0 \right] = \exp \left(-\frac{\|t\|^2}{2X_0^2} \right).$$

Using (3), the Fourier transform of the distribution of Y is thus $\exp(-\|t\|)$.

sub- σ -algebra \mathcal{C} of \mathcal{A} . Prove that if \mathcal{D} is the σ -algebra generated by $\mathcal{B} \cup \mathcal{C}$, then

$$\mathbf{E}[X|\mathcal{D}] = \mathbf{E}[X|\mathcal{B}].$$

METHOD. Prove the assertion first for square integrable X .

SOLUTION. If $X \in L^2(\mathcal{A})$, we must show that $Y = X - \mathbf{E}(X|B)$ is orthogonal not only to the subspace $L^2(\mathcal{B})$ of $L^2(\mathcal{A})$ but also to $L^2(\mathcal{D})$. For this, it suffices to show that Y is orthogonal to a dense subspace of $L^2(\mathcal{D})$. Since \mathcal{D} is generated by $\{B \cap C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$, a dense subspace of $L^2(\mathcal{D})$ is clearly formed by the set of those $Z \in \mathcal{D}$ for which there exist (i) a \mathcal{B} -measurable partition (B_1, \dots, B_n) of Ω , (ii) a \mathcal{C} -measurable partition (C_1, \dots, C_m) of Ω , and (iii) $(a_{ij})_{i=1}^n_{j=1}^m$ such that $Z = \sum_{i,j} a_{ij} \mathbf{1}_{B_i} \mathbf{1}_{C_j}$. It remains to show that $\mathbf{E}(YZ) = 0$. But

$$\begin{aligned} \mathbf{E}[YZ] &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B)) \mathbf{1}_{B_i} \mathbf{1}_{C_j}] \\ &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B)) \mathbf{1}_{B_i}] P[C_j], \end{aligned}$$

since Y is independent of \mathcal{C} . By definition, $\mathbf{E}(Y \mathbf{1}_{B_i}) = 0$. Hence $\mathbf{E}(YZ) = 0$.

Problem IV-36. If X and Y are integrable random variables such that $\mathbf{E}[X|Y] = Y$ and $\mathbf{E}[Y|X] = X$, show that $X = Y$ a.s.

METHOD. Show that, for fixed x ,

$$(i) \quad 0 \leq \int_{\{Y \leq x \leq X\}} (X - Y) dP = \int_{\{x < X \text{ and } x < Y\}} (Y - X) dP,$$

and conclude by symmetry that both sides of the equation are zero. Then use Problem I-13.

SOLUTION. Since $\mathbf{E}[X|X] = X$, we can write $\mathbf{E}[X - Y|X] = 0$; thus, for every Borel set A of \mathbf{R} ,

$$(ii) \quad \int_{X \in A} (X - Y) dP = 0.$$

Setting $A = \{X > x\} = \{Y \leq x < X\} \cup \{x < X \text{ and } x < Y\}$, (ii) implies (i). The positivity of the left-hand side is clear. Interchanging the roles of X and Y , the same reasoning gives

$$0 \leq \int_{\{x < X \text{ and } x < Y\}} (X - Y) dP.$$

This is the desired formula, since $C_n^k + C_n^{k-1} = C_{n+1}^k$.

(3) $(\exp(td))(\varphi)(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi^{(n)}(x) = \varphi(t+x)$ by Taylor's formula. It follows from (2) that

$$\begin{aligned} (\exp t(d+\rho))(\varphi)(x) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (d+\rho)^n \varphi(x) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{t^k}{k!} F_k(\rho) \frac{t^{n-k}}{(n-k)!} \varphi^{(n-k)}(x). \end{aligned}$$

But, by the definition of the F_k and formula V-1.4.2,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} F_k(x) = \exp \left[\frac{t^2}{2} + tx \right].$$

Thus, using the formula for the Cauchy product of two power series,

$$\begin{aligned} (\exp t(d+\rho))(\varphi)(x) &= \left(\sum_{n=0}^{\infty} \frac{t^k}{k!} F_k(\rho) \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi^{(n)}(x) \right) \\ &= \left(\exp \left(\frac{t^2}{2} + t\rho \right) \tau_t \right) (\varphi)(x). \end{aligned}$$

In particular, if $\lambda = 1$ then $\rho(\varphi) = x\varphi(x)$ and

$$(\exp t(d+x))(\varphi)(x) = \exp \left(\frac{t^2}{2} + tx \right) \varphi(t+x).$$

REMARK. The result of (2) is due to Viskov¹; that of (3) is due to Ville².

Problem V-4. Let X and Y be independent random variables with the same distribution $\nu_1(dx) = \exp(-\frac{x^2}{2}) \frac{dx}{\sqrt{2\pi}}$. Let $g : \mathbf{R} \rightarrow [0, +\infty)$ be a measurable function and let $Z = X + Y\sqrt{g(X)}$. Assume that Z has a normal distribution. Cantelli conjectured in 1917 that g is then constant almost everywhere; this is still unproved in 1994.

(1) Let $g_0 = \mathbf{E}(g(X))$. For all real t , compute $\mathbf{E}(\exp tZ)$ as a function of g_0 . Prove that $\exp(\alpha g) \in L^2(\nu_1)$ for all $\alpha > 0$.

METHOD. Use the Schwarz inequality.

(2) Let $\{g_n\}_{n=0}^{\infty}$ be the sequence of real numbers such that $g(x) = \sum_{n=0}^{\infty} g_n \frac{H_n(x)}{n!}$ in the $L^2(\nu_1)$ sense. By considering $\mathbf{E}(Z^3)$ and $\mathbf{E}(Z^4)$, show that $g_1 = 0$ and $-2g_2 = \sum_{n=2}^{\infty} \frac{g_n^2}{n!}$.

¹O. Viskov, *Theory of Probability and Its Applications*, Vol. 30, n. 1 (1984), 141–143.

²J. Ville, *Comptes Rendus Acad. des Sc.* 221 (1945), 529–539.