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Exercises and Solutions Manual for

Integration and Probability

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Contents

(1) Use Problem III-2 to show that the distribution of Y is independent of ν.
(2) From now on, assume that the $\{X_j\}_{j=0}^n$ are independent, with the same distribution μ and with Fourier transform $\exp(-\frac{t^2}{2})$. Show, using Problem III-1, that μ must be invariant under every orthogonal matrix.
(3) If $a \in \mathbf{R}$, compute the integral

$$I(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\left(x^2 + \frac{a^2}{x^2}\right)\right] dx$$

by using the following fact from Problem II-12(1):

$$\int_{-\infty}^{+\infty} f\left(x - \frac{|a|}{x}\right) dx = \int_{-\infty}^{+\infty} f(y)dy \quad \textit{for every } f \textit{ integrable on } \mathbf{R}.$$

(4) By first conditioning with respect to X_0 (see Problem IV-34), compute the Fourier transform of the distribution of Y.
(5) Using Problem IV-11, find the distribution of $\|Y\|^2$. Derive the density of Y from this, by observing that the distribution of Y is invariant under every orthogonal matrix in O_d and using Problem III-3.

SOLUTION. (1) By Problem III-2(4), if $U = (U_0, \dots, U_d)$ is a random variable concentrated on the unit sphere S_d of $\mathbf{R}^{d+1}$, independent of $\|X\|$ and with rotation-invariant distribution σ, then X and $U\|X\|$ have the same distribution. Hence Y and $(\frac{U_1}{U_0}, \frac{U_2}{U_0}, \dots, \frac{U_d}{U_0})$ have the same distribution, which proves the result.
(2) $\mathbf{E}[\exp(i\sum_{j=0}^d t_j X_j)] = \exp(-\frac{1}{2}\|t\|^2)$. Problem III-1 gives the result.
(3) Set $f(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})$. Then

$$f\left(x - \frac{|a|}{x}\right) = e^{+|a|} \exp\left(-\frac{1}{2}\left(x^2 + \frac{a^2}{x^2}\right)\right).$$

Hence $I(a) = e^{-|a|} I(0) = e^{-|a|}$.

$$(4) \qquad \mathbf{E}\left[\exp\left(i\sum_{j=1}^d t_j \frac{X_j}{X_0}\right)\right] = \mathbf{E}\left[\mathbf{E}\left[\exp\left(i\sum_{j=1}^d t_j \frac{X_j}{X_0}\right) \mid X_0\right]\right].$$

But, by (2),

$$\mathbf{E}\left[\exp\left(i\sum_{j=1}^d t_j \frac{X_j}{X_0}\right) \mid X_0\right] = \exp\left(-\frac{\|t\|^2}{2X_0^2}\right).$$

Using (3), the Fourier transform of the distribution of Y is thus $\exp(-\|t\|)$.

sub-σ-algebra $\mathcal{C}$ of $\mathcal{A}$. Prove that if $\mathcal{D}$ is the σ-algebra generated by $\mathcal{B} \cup \mathcal{C}$, then

$$\mathbf{E}[X|\mathcal{D}] = \mathbf{E}[X|\mathcal{B}].$$

METHOD. Prove the assertion first for square integrable X.

SOLUTION. If $X \in L^2(\mathcal{A})$, we must show that $Y = X - \mathbf{E}(X|B)$ is orthogonal not only to the subspace $L^2(\mathcal{B})$ of $L^2(\mathcal{A})$ but also to $L^2(\mathcal{D})$. For this, it suffices to show that Y is orthogonal to a dense subspace of $L^2(\mathcal{D})$. Since $\mathcal{D}$ is generated by $\{B \cap C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$, a dense subspace of $L^2(\mathcal{D})$ is clearly formed by the set of those $Z \in \mathcal{D}$ for which there exist (i) a $\mathcal{B}$-measurable partition $(B_1, \ldots, B_n)$ of Ω, (ii) a $\mathcal{C}$-measurable partition $(C_1, \ldots, C_m)$ of Ω, and (iii) $(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ such that $Z = \sum_{i,j} a_{ij} \mathbf{1}_{B_i} \mathbf{1}_{C_j}$. It remains to show that $\mathbf{E}(YZ) = 0$. But

$$\begin{aligned} \mathbf{E}[YZ] &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B))\mathbf{1}_{B_i}\mathbf{1}_{C_j}] \\ &= \sum_{i,j} a_{ij} \mathbf{E}[(X - \mathbf{E}(X|B))\mathbf{1}_{B_i}] P[C_j], \end{aligned}$$

since Y is independent of $\mathcal{C}$. By definition, $\mathbf{E}(Y\mathbf{1}_{B_i}) = 0$. Hence $\mathbf{E}(YZ) = 0$.

Problem IV-36. *If X and Y are integrable random variables such that $\mathbf{E}[X|Y] = Y$ and $\mathbf{E}[Y|X] = X$, show that $X = Y$ a.s.*

METHOD. Show that, for fixed x,

$$(i) \qquad 0 \le \int_{\{Y \le x \le X\}} (X - Y)dP = \int_{\{x < X \text{ and } x < Y\}} (Y - X)dP,$$

and conclude by symmetry that both sides of the equation are zero. Then use Problem I-13.

SOLUTION. Since $\mathbf{E}[X|X] = X$, we can write $\mathbf{E}[X - Y|X] = 0$; thus, for every Borel set A of $\mathbf{R}$,

$$(ii) \qquad \int_{X \in A} (X - Y)dP = 0.$$

Setting $A = \{X > x\} = \{Y \le x < X\} \cup \{x < X \text{ and } x < Y\}$, (ii) implies (i). The positivity of the left-hand side is clear. Interchanging the roles of X and Y, the same reasoning gives

$$0 \le \int_{\{x < X \text{ and } x < Y\}} (X - Y)dP.$$

This is the desired formula, since $C_n^k + C_n^{k-1} = C_{n+1}^k$.
(3) $(\exp(td))(\varphi)(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\varphi^{(n)}(x) = \varphi(t+x)$ by Taylor's formula. It follows from (2) that

$$\begin{aligned}(\exp\ t(d+\rho))(\varphi)(x) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}(d+\rho)^n\varphi(x)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t^k}{k!}F_k(\rho)\frac{t^{n-k}}{(n-k)!}\varphi^{(n-k)}(x).\end{aligned}$$

But, by the definition of the F_k and formula V-1.4.2,

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}F_k(x) = \exp\left[\frac{t^2}{2}+tx\right].$$

Thus, using the formula for the Cauchy product of two power series,

$$\begin{aligned}(\exp\ t(d+\rho))(\varphi)(x) &= \left(\sum_{n=0}^{\infty}\frac{t^k}{k!}F_k(\rho)\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\varphi^{(n)}(x)\right)\\ &= \left(\exp(\tfrac{t^2}{2}+t\rho)\tau_t\right)(\varphi)(x).\end{aligned}$$

In particular, if $\lambda = 1$ then $\rho(\varphi) = x\varphi(x)$ and

$$(\exp t(d+x))(\varphi)(x) = \exp(\frac{t^2}{2}+tx)\varphi(t+x).$$

REMARK. The result of (2) is due to Viskov[1]; that of (3) is due to Ville[2].

Problem V-4. *Let X and Y be independent random variables with the same distribution $\nu_1(dx) = \exp(-\frac{x^2}{2})\frac{dx}{\sqrt{2\pi}}$. Let $g : \mathbf{R} \to [0, +\infty)$ be a measurable function and let $Z = X + Y\sqrt{g(X)}$. Assume that Z has a normal distribution. Cantelli conjectured in 1917 that g is then constant almost everywhere; this is still unproved in 1994.*
(1) Let $g_0 = \mathbf{E}(g(X))$. For all real t, compute $\mathbf{E}(\exp\ tZ)$ as a function of g_0. Prove that $\exp(\alpha g) \in L^2(\nu_1)$ for all $\alpha > 0$.
METHOD. Use the Schwarz inequality.

(2) Let $\{g_n\}_{n=0}^{\infty}$ be the sequence of real numbers such that $g(x) = \sum_{n=0}^{\infty} g_n \frac{H_n(x)}{n!}$ in the $L^2(\nu_1)$ sense. By considering $\mathbf{E}(Z^3)$ and $\mathbf{E}(Z^4)$, show that $g_1 = 0$ and $-2g_2 = \sum_{n=2}^{\infty}\frac{g_n^2}{n!}$.

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[1] O. Viskov, *Theory of Probability and Its Applications*, Vol. 30, n. 1 (1984), 141–143.
[2] J. Ville, *Comptes Rendus Acad. des Sc. 221* (1945), 529–539.