Solutions Manual

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for the book

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21. $(-i)[(1-i)z_1+3z_2]+(1-i)[iz_1+(1+2i)z_2]$
$= -i(2-3i)+(1-i)(1)$
$\Longrightarrow z_2 = \dfrac{-2-3i}{3-2i} = -i \Longrightarrow z_1 = 1+i$

22. $0 = z^4-16 = (z-2)(z+2)(z-2i)(z+2i) \Longrightarrow z = 2, -2, 2i, -2i$

23. Suppose $z = a+bi$.

$$\mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}\left(\frac{a-ib}{a^2+b^2}\right) = \frac{a}{a^2+b^2} > 0$$

whenever $a > 0$.

24. Suppose $z = a+bi$.

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{z}\right) &= \mathrm{Im}\left(\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i\right)\\ &= -\frac{b}{a^2+b^2} < 0 \text{ whenever } b > 0.\end{aligned}$$

25. Let $z_1 = a+bi$ and $z_2 = c+di$. The hypotheses specify that $a+c<0$, $b+d=0$, $ac-bd<0$, and $ad+bc=0$.

$b=0 \Longrightarrow d=0 \Longrightarrow z_1$ and z_2 are real.

$b \neq 0 \Longrightarrow d=-b$ and $ad+bc = a(-b)+bd = -b(a-c) = 0$

$\Longrightarrow a=c$, a contradiction of the fact that $z_1z_2<0$.

26. By induction: The case when $n=1$ is obvious. Assume $\mathrm{Re}\left(\sum_{j=1}^m z_j\right) = \sum_{j=1}^m \mathrm{Re}(z_j)$ for all positive integers $m<n$

$$\begin{aligned}\mathrm{Re}\left(\sum_{j=1}^n z_j\right) &= \mathrm{Re}\left(\sum_{j=1}^{n-1} z_j + z_n\right)\\ &= \sum_{j=1}^{n-1}\mathrm{Re}(z_j)+\mathrm{Re}(z_n)\\ &= \sum_{j=1}^{n}\mathrm{Re}(z_j)\end{aligned}$$

The corresponding result for the imaginary parts follows by replacing "Re" by "Im" in the above proof.

20. . Define subroutines called sum, diff, prod, and quot based on exercise 31, section 1.1. Also define subroutines called polar and rectangular based on exercise 24, section 1.3. Define compsqrt(x, y) as follows:
Input x, y
Set (r, t)=polar(x, y)
Set new$r$=sqrt(r), new$t$=$t/2$
Set (new$x$, new$y$)=rectangular(new$r$,new$t$)
Output (new$x$, new$y$)
Stop

Now the quadratic formula program can be written.

Input ar, ai, br, bi, cr, ci
Set (discrim r, discrim i) = prod(br, bi, br, bi)$-4 *$ prod (ar, ai, cr, ci)
Set (toproot$r$, toproot$i$)=compsqrt(discrim$r$, discrim$i$)
Set ($z1r$, $z1i$)=quot($-br +$ toproot$r$, $-bi +$ toproot$i$, $2 * ar$, $2 * ai$)
Set ($z2r$, $z2i$)=quot($-br -$ toproot$r$, $-bi -$ toproot$i$, $2 * ar$, $2 * ai$)
Print "One solution is $(x, y) =$"; ($z1r$, $z1i$); "which is $(r, t) =$";
polar ($z1r$, $z1i$)
Print "The other solution is $(x, y) =$"; ($z2r$, $z2i$); "which is $(r, t) =$";
polar($z2r$, $z2i$)
Stop

21. (a) $\pm(3+i)$ (b) $\pm(3+2i)$ (c) $\pm(5+i)$
(d) $\pm(2-i)$ (e) $\pm(1+3i)$ (f) $\pm(3-i)$

EXERCISES 1.6: Planar Sets

1. Let z_1 be in the neighborhood $|z - z_0| < \rho$ and let $R = \rho - |z_1 - z_0|$. Choose a point ω in $|z - z_1| < R$. Then

$$\begin{aligned} |z_0 - \omega| &= |z_0 - z_1 + z_1 - \omega| \\ &\leq |z_0 - z_1| + |z_1 - \omega| \\ &< |z_0 - z_1| + R = \rho \end{aligned}$$

so z_1 is an interior pont of $|z - z_0| < \rho$ and the neighborhood is an open set.

9. a. $2-3i$

   b. $\pm i$

   c. $\dfrac{-1\pm i\sqrt{15}}{2}$

   d. $\dfrac{1}{2}, 1$

10. $\displaystyle\lim_{\Delta z\to 0}\frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$

$$= \lim_{\Delta z\to 0}\frac{(z_0+\Delta z)(\overline{z}_0+\overline{\Delta z})-z_0\overline{z}_0}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\left(\overline{z}_0+\frac{\overline{\Delta z}}{\Delta z}z_0+\overline{\Delta z}\right)=\begin{cases}\overline{z_0}+z_0 & \text{if } \Delta z=\Delta x\\ \overline{z_0}-z_0 & \text{if } \Delta z=i\Delta y\end{cases}$$

If $z_0=0$, then the difference quotient is

$$\lim_{\Delta z\to 0}(0+0+\overline{\Delta z})=0.$$

11. a. nowhere analytic

    b. nowhere analytic

    c. analytic except at $z=5$

    d. everywhere analytic

    e. nowhere analytic

    f. analytic except at $z=0$

    g. nowhere analytic

    h. nowhere analytic

12. The case when $n=1$ is trivial. Assume that the result holds for all positive integers less than or equal to n and define $Q(z)=P(z)(z-z_{n+1})$. Since $Q'(z)=P'(z)(z-z_{n+1})+P(z)$, it follows that

$$\frac{Q'(z)}{Q(z)}=\frac{P'(z)}{P(z)}+\frac{1}{z-z_{n+1}}=\frac{1}{z-z_1}+\frac{1}{z-z_2}+\cdots+\frac{1}{z-z_{n+1}}$$

17. If $w = f(z)$ is any branch of $\log z$ analytic on a domain D, then $e^w = z$. For $z_0 \in D$,

$$\lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\displaystyle\lim_{z\to z_0} \frac{z - z_0}{w - w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}.$$

18. Let $G(z)$ be another branch of $\log z$ analytic on D. Then $G'(z) - F'(z) = 0$, so $G(z) = F(z) + c$. Since the imaginary part of each branch has to be a value of $\arg z$, the constant c must be a multiple of $2\pi i$. Thus $G(z) = F(z) + 2k\pi i$ for some value of $k = 0, \pm 1, \ldots$.

19. Define $\log z = \operatorname{Log}|z| + i\theta$ with

$$\theta = \begin{cases} \text{the value of } \arg z \text{ between } \pi/2 \text{ and } 2\pi \\ \qquad \text{for } z \text{ in quad. II, III, or IV} \\ \text{the value of } \arg z \text{ between } 0 \text{ and } \pi/2 \\ \qquad \text{for } z \text{ in quad. I above the half parabola} \\ \text{the value of } \arg z \text{ between } 2\pi \text{ and } 5\pi/2 \\ \qquad \text{for } z \text{ in quad. I below the half parabola} \end{cases}$$

To make this explicit, one could find θ as a function of r on the half parabola $y = \sqrt{x}$.

$$\theta = \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) = \operatorname{Tan}^{-1}\left(\frac{1}{y}\right)$$

$$= \operatorname{Tan}^{-1}\left(\frac{1}{r\sin\theta}\right) = \operatorname{Tan}^{-1}\sqrt{\frac{2}{\sqrt{1+4r^2}-1}}$$

20. Define subroutines called radius and argument based on Exercise 24, Section1.3.

a. INPUT x, y
Step1 If $x = 0$ and $y = 0$, go to step 6
Step2 If $x < 0$ and $y = 0$, set logarithm= (log($-x$), pi)
Step3 Else set logarithm = (log(radius(x, y)), argument ($x, y,$))
Step4 Print "logarithm is "; logarithm
Step5 Go to step7
Step6 Print "undefined"
Step7 Stop

$$e^w = \frac{2z + (4z^2 - 4)^{1/2}}{2} = z + (z^2 - 1)^{1/2} \Rightarrow$$
$$w = \log[z + (z^2 - 1)^{1/2}]$$

14. Choose a branch of the square root and a branch of the logarithm.

$$\begin{aligned}
\frac{d}{dz}\left(\sinh^{-1} z\right) &= \frac{d}{dz}\left\{\log[z + (z^2+1)^{1/2}]\right\} \\
&= \frac{1 + z(z^2+1)^{-1/2}}{z + (z^2+1)^{1/2}} \\
&= \frac{1}{(z^2+1)^{1/2}} \cdot \frac{(z^2+1)^{1/2} + z}{z + (z^2+1)^{1/2}} \\
&= \frac{1}{(z^2+1)^{1/2}} \quad z \neq \pm i
\end{aligned}$$

15. a. $i \exp\left[\frac{1}{2}\text{Log}(1 - z^2)\right]$

b. $z \exp\left[\frac{1}{2}\text{Log}(1 + 4/z^2)\right]$

c. $z^2 \exp\left[\frac{1}{2}\text{Log}(1 - 1/z^4)\right]$

d. $z \exp\left[\frac{1}{3}\text{Log}(1 - 1/z^3)\right]$

16. Choose a branch of $\log z$ that is analytic at c.

$$\begin{aligned}
\frac{d}{dz}c^z &= \frac{d}{dz}e^{z\log c} \\
&= \log c\, e^{z\log c} \\
&= (\log c)\, c^z \text{ for all } z
\end{aligned}$$

17. Set $w = \sec^{-1} z$. Then

$$z = \sec w = \frac{2}{e^{iw} + e^{-iw}}$$

$$ze^{iw} + ze^{-iw} = 2$$

Here $G(z)$ is zero inside Γ, while on the boundary $|g(\zeta)| = 1$, so that $\lim_{z \to \zeta} G(z) \neq g(\zeta)$. This does not violate Cauchy's formula because $g(\zeta) = \frac{1}{\zeta}$ is not analytic on any simply connected domain containing $\Gamma : |\zeta| = 1$.

14. a. $\cos(2 + 3i)$

    b. 0

15. By Theorem 15, $G(z) = \dfrac{1}{2\pi i} \oint_{|\zeta|=1} \dfrac{f(\zeta)}{\zeta(\zeta - z)} d\zeta$ is analytic for all z not on $|z| = 1$.

$$G(0) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta^2} d\zeta = f'(0) = F(0) \text{ (Theorem 19)}$$

$$\text{For } z \neq 0, G(z) = \frac{1}{z}\left[\frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta} d\zeta\right]$$

$$= \frac{1}{z}(f(z) - f(0)) = F(z)$$

Therefore $G(z) = F(z)$ for any z in $|z| < 1$, and $F(z)$ is analytic on $|z| \leq 1$.

16. a. By Theorem 16, $f'(z)$ is analytic in D. By Theorem 3, Section 2.3 the quotient of analytic functions is analytic when the denominator is not zero.

    b. Theorem 10(13), Section 4.4

    c. $H'(z) = \dfrac{f'(z)}{f(z)}$, so

$$\frac{d}{dz}\left[f(z)e^{-H(z)}\right] = f'(z)e^{-H(z)} + f(z)e^{-H(z)}\left[\frac{-f'(z)}{f(z)}\right] = 0.$$

    Thus $f(z)e^{-H(z)} \equiv c$, since its derivative is zero. (Theorem 6, Section 2.4). Thus $f(z) = ce^{H(z)}$.

    d. $H(z) + \alpha$ is an analytic function in D and $f(z) = e^{H(z)+\alpha}$, so $H(z) + \alpha$ is a branch of $\log f(z)$.

For w in the right half-plane, $T_0(w)$ has a positive real part. Similarly, $T_k(w)$ has a positive real part for $k = 1, 2, 3, \ldots$ Thus $T_k(RHP) \subseteq RHP$.

To see more precisely how T_0 maps the right half-plane, calculate $T_0(0), T_0(i)$, and $T_0(\infty)$. It is straightforward to show that each of these points lies in the right half-plane within $\left|\zeta - \frac{1}{2}\right| \le \frac{1}{2}$ (and $T_0(\infty) = 0$). It follows that T_0 maps the right half-plane to an open disk inside $\left|\zeta - \frac{1}{2}\right| = \frac{1}{2}$. Then any number of applications of T_k maps the RHP inside the RHP, and $T_0 \circ T_1 \circ T_2 \circ \cdots \circ T_{n-2} \circ T_{n-1}(RHP) \subseteq T_0(RHP) \subseteq \left|\zeta - \frac{1}{2}\right| < \frac{1}{2}$

24. $\dfrac{Q(z)}{P(z)} = T_0 \circ T_1 \circ \cdots \circ T_{n-1}(0)$ as defined in Problem 21, so $Q(z)/P(z)$ maps the closed right half-plane into $\left|\zeta - \frac{1}{2}\right| \le \frac{1}{2}$. Thus all the poles of $Q(z)/P(z)$ (corresponding to zeros of $P(z)$) are in the left half-plane.

25. $\dfrac{Q(z)}{P(z)} = \dfrac{3z^2 + 6}{z^3 + 3z^2 + 6z + 6} = \dfrac{3}{z + 3 + \frac{4z}{z^2+2}} = \dfrac{3}{z + 3 + \frac{4}{z+2/z}}$

By Problem 22, $P(z)$ has all its zeros in the left half-plane.

EXERCISES 7.5: The Schwarz-Christoffel Transformation

1. At the corner $w_1 = -1$ the polygon takes a right turn of θ_1 with $\theta_1 \to \pi$. For any x_1 chosen as the preimage of w_1,

$$f(z) = \lim_{\theta \to \pi} A \int_0^z (\zeta - x_1)^{\theta_1/\pi} d\zeta + B$$

$$= A(z - x_1)^2 + B$$

(These are not the same A and B, but they are still constants that we have yet to determine, so we will not create new notation like A' and B' in this and the following problems.)
$f(x_1) = -1 \Longrightarrow B = -1$
$f(\pm\infty) = -\infty \Longrightarrow A < 0$
$f(z) = A(z - x_1)^2 - 1$ with $A < 0$