

# Instructor's Solutions Manual

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# Differential Equations

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With Boundary  
Value Problems

SECOND EDITION

POLKING

BOGGESS

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## 64 Chapter 2 First-Order Equations

that  $y_1(t) = -1$  and  $y_2(t) = 1$  are both solutions to the differential equation. If  $y$  is a solution and satisfies  $y(1) = 0$ , then  $y_1(1) < y(1) < y_2(1)$ . By the uniqueness theorem we must have  $y_1(t) < y(t) < y_2(t)$  for all  $t$  for which  $y$  is defined. Hence  $-1 < y(t) < 1$  for all  $t$  for which  $y$  is defined.

30. Notice that  $x_1(t) = 0$  and  $x_2(t) = 1$  are solutions to the same differential equation with initial values  $x_1(0) = 0 < 1/2 = x(0) < 1 = x_2(0)$ . The right hand side of the differential equation,  $f(t, x) = (x^3 - x)/(1 + t^2x^2)$ , and

$$\frac{\partial f}{\partial x} = \frac{(3x^2 - 1)(1 + t^2x^2) - 2t^2x(x^3 - x)}{(1 + t^2x^2)^2},$$

are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for  $x$ ,  $x_1$ , and  $x_2$  cannot cross. Hence we must have  $0 = x_1(t) < x(t) < x_2(t) = 1$  for all  $t$ .

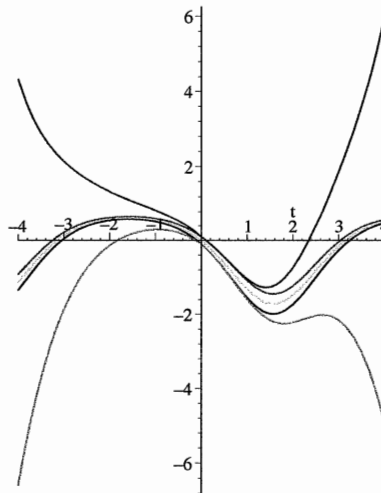
31. Notice that  $x_1(t) = t^2$  is a solution to the same differential equation with initial value  $x_1(0) = 0 < 1 = x(0)$ . The right hand side of the differential equation,  $f(t, x) = x - t^2 + 2t$  and  $\partial f/\partial x = 1$  are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for  $x$  and  $x_1$  cannot cross. Hence we must have  $t^2 = x_1(t) < x(t)$  for all  $t$ .

32. Notice that  $y_1(t) = \cos t$  is a solution to the same differential equation with initial value  $y_1(0) = 1 < 2 = y(0)$ . The right hand side of the differential equation,  $f(t, y) = y^2 - \cos^2 t - \sin t$  and  $\partial f/\partial y = 2y$  are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for  $y$  and  $y_1$  cannot cross. Hence we must have  $y(t) > y_1(t) = \cos t$  for all  $t$ .

—————x—————

## Section 2.8. Dependence of Solutions on Initial Conditions

1.  $x(0) = 0.8009$
2.  $x(0) = .9084$
3.  $x(0) = 0.9596$
4.  $x(0) = 0.9826$
5.  $x(0) = 0.7275$
6.  $x(0) = 0.72897$
7.  $x(0) = 0.7290106$
8.  $x(0) = 0.729011125$
9.  $x(0) = -3.2314$
10.  $x(0) = -3.23208$
11.  $x(0) = -3.2320923$
12.  $x(0) = -3.23092999999$
13. Ten! :-)
14.  $1 - e^{\sin t} - (1/10)e^{|t|} \leq y(t) \leq 1 - e^{\sin t} + (1/10)e^{|t|}$



## Chapter 4. Second-Order Equations

### Section 4.1. Definitions and Examples

1. Compare

$$y'' + 3y' + 5y = 3 \cos 2t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that  $p(t) = 3$ ,  $q(t) = 5$ , and  $g(t) = 3 \cos 2t$ . Hence, the equation is linear and inhomogeneous.

2. Divide both sides of  $t^2 y'' = 4y' - \sin t$  by  $t^2$ , then rearrange to obtain

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}.$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that  $p(t) = -4/t^2$ ,  $q(t) = 0$ , and  $g(t) = -(\sin t)/t^2$ . Hence, the equation is linear and inhomogeneous.

3. Expand  $t^2 y'' + (1 - y)y' = \cos 2t$  to obtain

$$t^2 y'' + y' - yy' = \cos 2t.$$

Note that the term  $yy'$  is nonlinear. Hence, this equation is nonlinear.

4. Divide both sides of  $ty'' + (\sin t)y' = 4y - \cos 5t$  by  $t$ , then rearrange to obtain

$$y'' + \frac{\sin t}{t} y' - \frac{4}{t} y = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that  $p(t) = (\sin t)/t$ ,  $q(t) = -4/t$ , and  $g(t) = -(\cos 5t)/t$ . Hence, the equation is linear and inhomogeneous.

5. In

$$t^2 y'' + 4yy' = 0,$$

note that the term  $4yy'$  is nonlinear. Hence, this equation is nonlinear.

6. Compare

$$y'' + 4y' + 7y = 3e^{-t} \sin t$$

with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that  $p(t) = 4$ ,  $q(t) = 7$ , and  $g(t) = 3e^{-t} \sin t$ . Hence, the equation is linear and inhomogeneous.

7. In

$$y'' + 3y' + 4 \sin y = 0$$

note that the term  $4 \sin y$  is nonlinear. Hence, this equation is nonlinear.

8. Divide both sides of  $(1 - t^2)y'' = 3y$  by  $1 - t^2$ , then rearrange the terms to obtain

$$y'' - \frac{3}{1 - t^2} y = 0.$$

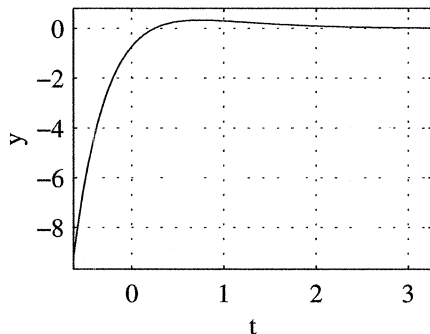
Compare with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that  $p(t) = 0$ ,  $q(t) = -3/(1 - t^2)$ , and  $g(t) = 0$ . Hence, the equation is linear and homogeneous.

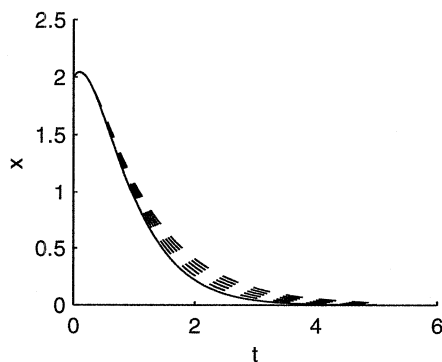
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$t$ .



Note that the graph crosses the  $t$ -axis exactly once. Finally, by picking initial conditions from the unshaded region, you will note that this solution also crosses the  $y$ -axis exactly once, but at  $t < 0$ .

20. (a) The system  $x'' + \mu x' + 4x = 0$  has characteristic equation  $\lambda^2 + \mu\lambda + 4 = 0$ . If  $\mu = 4$ , this becomes  $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ , and there is one repeated root, hence the critically damped case.
- (b) The critically damped solution (solid line in figure) approaches the  $t$ -axis faster than any of the other overdamped solutions.



An overdamped screen door will shut on its own without slamming. A critically damped door will shut as fast as possible without slamming.

21. (a) Suppose that  $mx'' + \mu x' + kx = 0$  is overdamped. We can write

$$x'' + \frac{\mu}{m}x' + \frac{k}{m}x = 0$$

$$x'' + 2cx' + \omega_0^2 x = 0,$$

where  $2c = \mu/m$  and  $\omega_0^2 = k/m$ . The system has characteristic equation  $\lambda^2 + 2c\lambda + \omega_0^2 = 0$  and zeros

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2} \quad \text{and}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

If we set  $\gamma = \sqrt{c^2 - \omega_0^2}$ , then

$$\lambda_1 = -c - \gamma \quad \text{and} \quad \lambda_2 = -c + \gamma,$$

and  $\lambda_2 - \lambda_1 = 2\gamma$ . If the system is overdamped, note that

$$c^2 - \omega_0^2 > 0$$

$$\left(\frac{\mu}{2m}\right)^2 > \frac{k}{m}$$

$$\mu^2 > 4mk$$

$$\mu > 2\sqrt{mk}.$$

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

The initial condition  $x(0) = 0$  gives  $0 = C_1 + C_2$  and  $C_1 = -C_2$ . Differentiating  $x(t)$ ,

$$x'(t) = C_1 \lambda_1 e^{\lambda_1 t} + C_2 \lambda_2 e^{\lambda_2 t},$$

and the initial condition  $x'(0) = v_0$  provides  $v_0 = C_1 \lambda_1 + C_2 \lambda_2$ . This system is easily solved for

$$C_1 = \frac{v_0}{\lambda_1 - \lambda_2} = -\frac{v_0}{2\gamma} \quad \text{and}$$

$$C_2 = \frac{-v_0}{\lambda_1 - \lambda_2} = \frac{v_0}{2\gamma},$$

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$C_1 = -3/5$  and  $C_2 = -11/20$ . Therefore, the solution is

$$y = e^t \left( -\frac{3}{5} \cos 2t - \frac{11}{20} \sin 2t \right) + \frac{3}{5} \cos t - \frac{3}{10} \sin t.$$

22. The homogeneous equation  $y'' + 4y' + 4y = 0$  has characteristic equation  $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$  and repeated root  $\lambda = -2$ . Thus the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{-2t}.$$

The particular solution  $y_p = at + b$  has derivatives  $y'_p = a$  and  $y''_p = 0$ , which when substituted in  $y'' + 4y' + 4y = 4 - t$ ,

$$\begin{aligned} 4a + 4(at + b) &= 4 - t \\ 4at + (4a + 4b) &= -t + 4. \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 4a &= -1 \\ 4a + 4b &= 4, \end{aligned}$$

which has solution  $a = -1/4$  and  $b = 5/4$ . Thus, the general solution is

$$y = (C_1 + C_2 t)e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

The initial condition  $y(0) = -1$  provides

$$-1 = C_1 + \frac{5}{4}.$$

Differentiate.

$$y' = C_2 e^{-2t} - 2e^{-2t}(C_1 + C_2 t) - \frac{1}{4}$$

The initial condition  $y'(0) = 0$  provides

$$0 = C_1 - 2C_1 - \frac{1}{4}.$$

Thus,  $C_1 = -9/4$  and  $C_2 = -17/4$  and

$$y = \left( -\frac{9}{4} - \frac{17}{4}t \right) e^{-2t} - \frac{1}{4}t + \frac{5}{4}.$$

23. The homogeneous equation  $y'' - 2y' + y = 0$  has characteristic equation  $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ , with repeated zero  $\lambda = 1$ . Thus, the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^t.$$

The particular solution  $y_p = at^3 + bt^2 + ct + d$  has derivatives

$$\begin{aligned} y'_p &= 3at^2 + 2bt + c \\ y''_p &= 6at + 2b, \end{aligned}$$

which when substituted in  $y'' - 2y' + y = t^3$ , rearranging, yields

$$at^3 + (-6a + b)t^2 + (6a - 4b + c)t + (2b - 2c + d) = t^3.$$

Thus,

$$\begin{aligned} a &= 1 \\ -6a + b &= 0 \\ 6a - 4b + c &= 0 \\ 2b - 2c + d &= 0, \end{aligned}$$

which has solution  $a = 1$ ,  $b = 6$ ,  $c = 18$ , and  $d = 24$ . Thus, the general solution is

$$y = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24.$$

The initial condition  $y(0) = 1$  gives  $1 = C_1 + 24$ . Differentiating,

$$y' = C_2 e^t + (C_1 + C_2 t)e^t + 3t^2 + 12t + 18.$$

The initial condition  $y'(0) = 0$  gives  $0 = C_2 + C_1 + 18$ . The system has solution  $C_1 = -23$  and  $C_2 = 5$ . Therefore, the solution is

$$y = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24.$$

24. The homogeneous equation  $y'' - 3y' - 10y = 0$  has characteristic equation  $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$  with zeros  $\lambda_1 = 5$  and  $\lambda_2 = -2$ . Thus, the homogeneous solution is

$$y_h = C_1 e^{5t} + C_2 e^{-2t}.$$

Thus, the forcing term of  $y'' - 3y' - 10y = 3e^{-2t}$  is a solution of the homogeneous equation. Substitute  $y_p = Ate^{-2t}$  and its derivatives

$$\begin{aligned} y'_p &= Ae^{-2t}(1 - 2t) \\ y''_p &= (-4 - 4t)Ae^{-2t} \end{aligned}$$



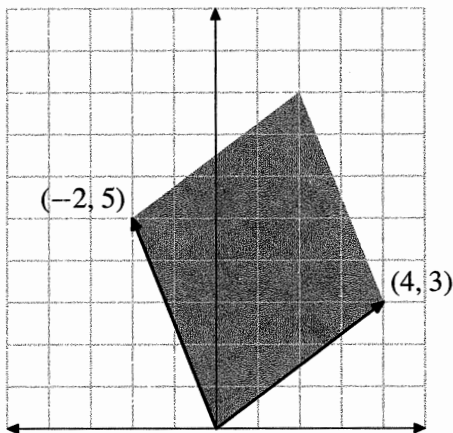
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The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 1 & 6 \\ 4 & 1 \end{vmatrix} = (1)(1) - (4)(6) \\ &= 1 - 24 = -23. \end{aligned}$$

Note that the determinant is the negative of the area.

4. Estimate the area by counting square units inside the parallelogram in

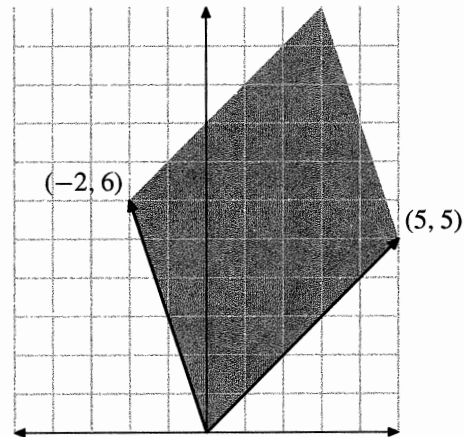


The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = (-2)(3) - (5)(4) \\ &= -6 - 20 = -26. \end{aligned}$$

Note that the determinant is the negative of the area.

5. Estimate the area by counting square units inside the parallelogram in



The determinant is

$$\begin{aligned} |\mathbf{v}_1, \mathbf{v}_2| &= \begin{vmatrix} 5 & -2 \\ 5 & 6 \end{vmatrix} = (5)(6) - (5)(-2) \\ &= 30 + 10 = 40. \end{aligned}$$

6. First note the determinant of  $A$ .

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To prove part (1), construct  $B$  by adding  $r$  times row 1 to row 2.

$$B = \begin{pmatrix} a & b \\ c + ra & d + rb \end{pmatrix}.$$

Then,

$$\begin{aligned} |B| &= \begin{vmatrix} a & b \\ c + ra & d + rb \end{vmatrix}, \\ &= a(d + rb) - b(c + ra), \\ &= ad - bc, \\ &= |A|. \end{aligned}$$

To prove part (2), craft  $B$  by swapping rows 1 and 2 of matrix  $A$ .

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

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or,

$$A\mathbf{y}_p + \mathbf{f} = \begin{pmatrix} (-3a_1 + 6a_2)t + (-3b_1 + 6b_2 + 3) \\ (-2a_1 + 4a_2)t + (-2b_1 + 4b_2 + 4) \end{pmatrix}.$$

Comparing coefficients of the polynomial entries (e.g.,  $0 = -3a_1 + 6a_2$  and  $a_1 = -3b_1 + 6b_2 + 3$ ), we get the following system.

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ a_1 + 3b_1 - 6b_2 &= 3 \\ a_2 + 2b_1 - 4b_2 &= 4 \end{aligned}$$

Solving,  $a_1 = -12$ ,  $a_2 = -6$ ,  $b_1 = 5 + 2b_2$ , with  $b_2$  free. Letting  $b_2 = 0$ , we get  $a_1 = -12$ ,  $a_2 = -6$ ,  $b_1 = 5$ , and  $b_2 = 0$ , providing the particular solution

$$\mathbf{y}_p = \begin{pmatrix} -12 \\ -6 \end{pmatrix} t + \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

17. If  $\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t}$ , then

$$\begin{aligned} \mathbf{y}'_p &= \mathbf{a}e^{-t} - (\mathbf{a}t + \mathbf{b})e^{-t} = [-\mathbf{a}t + (\mathbf{a} - \mathbf{b})]e^{-t} \\ &= \begin{pmatrix} -a_1t + (a_1 - b_1) \\ -a_2t + (a_2 - b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Next,

$$\begin{aligned} A\mathbf{y}_p + \mathbf{f} &= A(\mathbf{a}t + \mathbf{b})e^{-t} + \mathbf{f} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1t + b_1 \\ a_2t + b_2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \\ &= \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix} e^{-t}. \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} -a_1t + (a_1 - b_1) \\ -a_2t + (a_2 - b_2) \end{pmatrix} \\ = \begin{pmatrix} (a_1 + 2a_2)t + (b_1 + 2b_2 + 1) \\ (2a_1 + a_2)t + (2b_1 + b_2) \end{pmatrix}. \end{aligned}$$

Comparing coefficients of the polynomial entries in these vectors (e.g.,  $-a_1 = (a_1 + 2a_2)$ ) leads to the system

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_1 - 2b_1 - 2b_2 &= 1 \\ a_2 - 2b_1 - 2b_2 &= 0. \end{aligned}$$

The solution of this system is  $a_1 = 1/2$ ,  $a_2 = -1/2$ ,  $b_1 = -1/4 - b_2$ , where  $b_2$  is free. Choosing  $b_2 = -1/8$  provides the solution  $a_1 = 1/2$ ,  $a_2 = -1/2$ ,  $b_1 = -1/8$ , and  $b_2 = -1/8$ , so

$$\mathbf{y}_p = (\mathbf{a}t + \mathbf{b})e^{-t} = \begin{pmatrix} (1/2)t - 1/8 \\ (-1/2)t - 1/8 \end{pmatrix} e^{-t},$$

which is identical to that found in Example 9.15.

18. The current coming into the node at “a” must equal the current coming out of the same node. Hence,

$$i = i_1 + i_2. \quad (9.16)$$

Traversing (clockwise) the leftmost loop containing emf, resistor, and inductor, Kirchhoff’s voltage law provides

$$-30 + 10i + 0.02i'_1 = 0. \quad (9.17)$$

Traversing (clockwise) the outer loop containing emf, both resistors, and the far inductor,

$$-30 + 10i + 20i_2 + 0.04i'_2 = 0. \quad (9.18)$$

If we now substitute  $i = i_1 + i_2$  into equations (9.17) and (9.18), then a little algebra provides us with the following system (in the form  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ ).

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}' = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix} \quad (9.19)$$

A computer provides the following eigenvalue-eigenvector pairs for the coefficient matrix in equation (9.19):

$$-1000 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad -250 \rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence, the homogeneous solution is

$$\mathbf{x} = C_1 e^{-1000t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-250t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \quad (9.20)$$

Now for a particular solution, let’s try the form  $\mathbf{x}_p = (a_1, a_2)^T$ . Substituting this informed guess in equation (9.19), we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -500 & -500 \\ -250 & -750 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1500 \\ 750 \end{bmatrix}.$$

**694** Chapter 11 Series Solutions to Differential Equations

Choose  $y_1(x)$  with  $1 = y_1(0) = a_0$  and  $0 = y_1'(0) = a_1$ .

$$y_1(x) = 1 + \frac{1}{2 \cdot 1} x^2 - \frac{3}{4 \cdot 3 \cdot 2 \cdot 1} x^4 + \cdots = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} (3 \cdot 7 \cdot 11 \cdots (4n-5))}{(2n)!} x^{2n}$$

Choose  $y_2(x)$  with  $0 = y_2(0) = a_0$  and  $1 = y_2'(0) = a_1$ .

$$y_2(x) = x - \frac{1}{3 \cdot 2} x^3 + \frac{5}{5 \cdot 4 \cdot 3 \cdot 2} x^5 - \cdots = x - \frac{1}{6} x^3 + \sum_{n=2}^{\infty} \frac{(-1)^n (5 \cdot 9 \cdot 12 \cdots (4n-3))}{(2n+1)!} x^{2n+1}.$$

Note that the solutions are chosen so that

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the solutions are independent.

18. The coefficients of  $y'$  and  $y$  are  $p(x) = x$  and  $q(x) = 2$ , both polynomials and analytic at  $x = 0$ . Thus,  $x = 0$  is an ordinary point. According to Theorem 2.29, all solutions have radius of convergence  $R = \infty$ . We seek a solution of the form  $y = \sum_{n=0}^{\infty} a_n x^n$  with derivatives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting,

$$\begin{aligned} 0 = y'' + xy' + 2y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n. \end{aligned}$$

Shifting the index of the first term and noting that the second term

$$\sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n,$$

we write

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n+2) a_n] x^n.$$

Setting coefficients equal to zero,

$$(n+2)(n+1) a_{n+2} + (n+2) a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{-a_n}{n+1}, \quad n \geq 0.$$

Thus,

$$a_2 = -a_0, \quad a_4 = \frac{-a_2}{3} = \frac{a_0}{3}, \quad a_6 = \frac{-a_4}{5} = \frac{-a_0}{5 \cdot 3}, \quad \text{and} \quad a_8 = \frac{-a_6}{7} = \frac{a_0}{7 \cdot 5 \cdot 3}.$$

## Section 13.2. Separation of Variables for the Heat Equation

1. The thermal diffusivity of gold is  $k = 1.18 \text{ cm}^2/\text{sec}$ . We will let the unit of length be centimeters, so  $L = 50$ . The boundary conditions are  $u(0, t) = 0$  and  $u(50, t) = 0$ , so the steady-state temperature is 0. Hence the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L),$$

where the coefficients are

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L 100 \sin(n\pi x/L) dx \\ &= \frac{200}{n\pi} [1 - \cos(n\pi)] \\ &= \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus the solution is given by

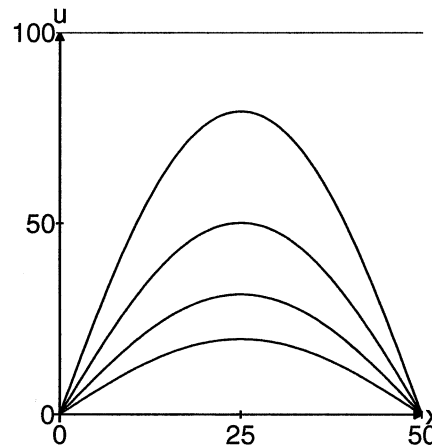
$$u(x, t) = \sum_{p=0}^{\infty} \frac{400}{(2p+1)\pi} e^{-1.18 \times (2p+1)^2 \pi^2 t/2500} \times \sin\left(\frac{n\pi x}{50}\right).$$

For  $t = 100$ , the term for  $p = 1$  is bounded by 0.65. Hence one term of the series will suffice to estimate the temperature within  $1^\circ$ . Using just this one term, we solve

$$\frac{400}{\pi} e^{-1.18 \times \pi^2 t/2500} = 10$$

for  $t = 546$  sec. Hence we see that it will take about 546 sec for the temperature to drop below  $10^\circ\text{C}$ . The

temperature at 100 second intervals is plotted below.



2. The thermal diffusivity of aluminum is  $k = 0.84 \text{ cm}^2/\text{sec}$ . For  $t = 100$ , the term for  $p = 1$  is bounded by 2.14, while that for  $p = 2$  is bounded by 0.006. Hence two terms will suffice to compute the temperature for  $t = 100$  sec. Looking at the term for  $p = 0$ , we solve

$$\frac{400}{\pi} e^{-0.84 \times \pi^2 t/2500} = 10$$

for  $t = 767$  sec. For such a time, all of the other terms are extremely small, so we see that it will take about 767 sec for the temperature to drop below  $10^\circ\text{C}$ .

The thermal diffusivity of silver is  $k = 1.7 \text{ cm}^2/\text{sec}$ . For  $t = 100$ , the term for  $p = 1$  is bounded by 0.1. Hence one term will suffice to compute the temperature for  $t = 100$  sec. Looking at the term for  $p = 0$ , we solve

$$\frac{400}{\pi} e^{-1.7 \times \pi^2 t/2500} = 10$$

for  $t = 379$  sec. For such a time, all of the other terms are extremely small, so we see that it will take about 379 sec for the temperature to drop below  $10^\circ\text{C}$ .