COMPLEX VARIABLES WITH APPLICATIONS

Third Edition

By A. David Wunsch
University of Massachusetts Lowell

SOLUTIONS MANUAL

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The bulky volume you are holding represents the solutions to all the problems in the 3rd edition of my textbook *Complex Variables with Applications*. Both Michael Brown and I have separately worked through all the solutions, but I can say with overwhelming confidence that in spite of this redundancy there are some remaining errors. Please tell me of any that you find. My e mail address is David_Wunsch@UML.edu (note the underscore). Those preferring an older medium of communication may write to me at the Electrical and Computer Engineering Dept. University of Massachusetts Lowell, Lowell, MA 01854. I promise to acknowledge all e-mail and postal mail that I receive. I would also appreciate learning of any errors in the textbook itself.

I plan to post corrections to both the book and this manual at the web address http://faculty.uml.edu/awunsch/Wunsch_Complex_Variables/

This manual has been written primarily for college faculty who are teaching from my text. Whether it is to be made freely available to students- perhaps at the school library- is a matter I leave up to each individual instructor. Notice however, that there is little point in assigning the textbook problems involving computer programming if students already have the MATLAB code supplied in this manual. Regarding this code, I must assert that I am not a professional programmer and I'm certain that in many cases the reader will find more efficient ways of solving the same problem.

Finally, I must apologize for the idiosyncrasies of the handwriting. They are my own and not to blamed on Mr. Brown.

A. David Wunsch
Belmont, Massachusetts
July 8, 2004

$$e^{i\pi}+1=0$$

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23)

$u = u(r(x,y), \theta(x,y))$

Thus $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\Big|_\theta \frac{\partial r}{\partial x}\Big|_y + \frac{\partial u}{\partial \theta}\Big|_r \frac{\partial \theta}{\partial x}\Big|_y$ [1]

$\frac{\partial u}{\partial y}$ (same as [1] but swap x and y

$\frac{\partial v}{\partial x}$ same as [1] but put v instead of u

$\frac{\partial v}{\partial y}$ same as [1] but put v instead of u, swap x and y.

(b) $r = \sqrt{x^2+y^2}$, $\frac{\partial r}{\partial x}\Big|_y = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\frac{\partial \theta}{\partial x}\Big|_y = \frac{1}{1+y^2/x^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} =$

$-\frac{y}{\sqrt{x^2+y^2}}\frac{1}{\sqrt{x^2+y^2}} = -\frac{\sin\theta}{r}$

$\frac{\partial r}{\partial y}\Big|_x = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$, $\frac{\partial \theta}{\partial y}\Big|_x = \frac{x}{\sqrt{x^2+y^2}}\frac{1}{\sqrt{x^2+y^2}} = \frac{\cos\theta}{r}$

Use the preceding equations in [1]

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\Big|_\theta \frac{\partial r}{\partial y}\Big|_x + \frac{\partial u}{\partial \theta}\Big|_r \frac{\partial \theta}{\partial y}\Big|_x$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$ similarly:

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$

(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ from (2.3-10 a)

Using result of part (b):

$\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$ [2]

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ from (2-3-10 b)

Appendix chap 3

1) $V(t) = \text{Re}\left[3e^{(1+2i)t}\right] =$

$\text{Re}\left[3e^t\left[\cos 2t + i\sin 2t\right]\right] = 3e^t \cos 2t$

2) $v(t) = \text{Re}\left[i\ e^{(-1+2i)t}\right] =$

$\text{Re}\left[ie^{-t}\left[\cos 2t + i\sin 2t\right]\right] = -e^{-t}\sin(2t)$

3) $\text{Re}\left[2e^{i\pi/3}e^{(1-2i)t}\right] = \text{Re}\left[2e^t e^{i(\frac{\pi}{3}-2t)}\right]$

$= 2e^t\cos\left[\pi/3 - 2t\right]$

4) $\text{Re}\left[i\ e^{-2it}\right] = \sin 2t$

5) $\text{Re}\left[(1+i)e^{2it}\right] = \cos 2t - \sin 2t$

or $\text{Re}\left[\sqrt{2}\,e^{i\pi/4}e^{2it}\right] = \text{Re}\left[\sqrt{2}\,e^{i\left[\frac{\pi}{4}+2t\right]}\right]$

$= \sqrt{2}\cos\left[\frac{\pi}{4}+2t\right]$

6) $\text{Re}\left[(1+e^{i\pi/4})e^{\left[e^{-i\pi/6}\right]t}\right] = \text{Re}\left[(1+e^{i\pi/4})e^{\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)t}\right]$

$= e^{\sqrt{3}/2\,t}\cos\frac{t}{2} + e^{\sqrt{3}/2\,t}\cos\left[\frac{\pi}{4}-\frac{t}{2}\right]$

7) $v(t) = \text{Re}\left[e^{(1+i)}\exp\left[t\,e^{i\pi/4}\right]\right] = \text{Re}\left[e^{1+i}\,e^{\frac{t}{\sqrt{2}}+\frac{it}{\sqrt{2}}}\right]$

$= \text{Re}\left[\exp\left[\frac{t}{\sqrt{2}}-\frac{t}{\sqrt{2}}\right]\ \exp i\left[\frac{t}{\sqrt{2}}+\frac{t}{\sqrt{2}}\right]\right]$

$= e^{\left(1+\frac{t}{\sqrt{2}}\right)}\cos\left[1+\frac{t}{\sqrt{2}}\right]$

14d] Use part (c). $\frac{z}{(z^2+1)(z-2)} = \frac{a}{z-i} + \frac{b}{z+i} + \frac{c}{z-2}$

$a = \lim_{z\to i} \frac{z}{(2z)(z-2)}\Big|_i = \frac{1}{(2)(i-2)}$, $b = \lim_{z\to -i} \frac{z}{(2z)(z-2)} = \frac{1}{(2)(-i-2)}$

$c = \lim_{z\to 2} \frac{z}{z^2+1} = \frac{2}{5}$

15] $\frac{z}{(z-1)(z+2)} = \frac{1/3}{(z-1)} + \frac{2/3}{z+2} = -\frac{1}{3}\left[1+z+z^2+z^3+\cdots\right] + \frac{1}{3}\frac{1}{1+z/2}$

$= -\frac{1}{3}\underset{|z|<1}{\left[1+z+z^2+z^3+\cdots\right]} + \frac{1}{3}\underset{|z|<2}{\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}\cdots\right]} = \sum_{n=0}^{\infty} c_n z^n$, $|z|<1$

where $c_n = \left[-\frac{1}{3} + \frac{1}{3}\frac{(-1)^n}{2^n}\right]$

16] $\frac{z}{(z+1)(z+2)} = \frac{-1}{(z+1)} + \frac{2}{(z+2)}$

$\frac{-1}{z+1} = \frac{-1}{z-1+2} = \frac{-1/2}{1+\frac{z-1}{2}} = -\frac{1}{2}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4}\cdots\right]$

$|z-1|<2$

$\frac{2}{z+2} = \frac{2}{(z-1)+3} = \frac{2/3}{1+\left(\frac{z-1}{3}\right)} = \frac{2}{3}\left[1-\left(\frac{z-1}{3}\right)+\left(\frac{z-1}{3}\right)^2\cdots\right]$

add the two previous series:

$\frac{z}{(z+1)(z+2)} = \sum_{n=0}^{\infty} c_n (z-1)^n$, $c_n = -\frac{1}{2}\left(-\frac{1}{2}\right)^n + \frac{2}{3}\left(-\frac{1}{3}\right)^n$

17] $\frac{1}{z^2} = \frac{1}{\left[(z-(1+i))+(1+i)\right]^2} = \frac{\frac{1}{(1+i)^2}}{\left[1+\frac{z-(1+i)}{(1+i)}\right]^2}$

$= \frac{1}{2i}\frac{1}{(1+w)^2}$ where $w = \frac{z-(1+i)}{1+i}$

$\frac{1}{(1+w)^2} = 1-2w+3w^2-4w^3\cdots$ (see text) Eq. 5.5-16(d), $|w|<1$,

$\frac{1}{2i}\frac{1}{(1+w)^2} = \frac{1}{z^2} = \sum_{n=0}^{\infty} c_n\left[z-(1+i)\right]^n$ where $c_n = \frac{1}{2i}\frac{(-1)^n(n+1)}{(1+i)^n}$

Since $|w|<1$, require $\frac{|z-(1+i)|}{|1+i|}<1$ or $|z-(1+i)|<\sqrt{2}$

32) for poles: $\sinh z = 2e^z$, $\frac{e^z - e^{-z}}{2} = 2e^z$

$e^z - e^{-z} = 4e^z$, $3e^z = -e^{-z}$, $3e^{2z} = -1$

$e^{2z} = -1/3$, $2z = \operatorname{Log}\frac{1}{3} + i[\pi + 2k\pi]$,

$z = \frac{1}{2}\operatorname{Log}\left[\frac{1}{3}\right] + i\left[\frac{\pi}{2} + k\pi\right] = \operatorname{Log}\frac{1}{\sqrt{3}} + i\left[\frac{\pi}{2} + k\pi\right]$

location of poles. Enclosed poles:

$z = \operatorname{Log}\frac{1}{\sqrt{3}} + i\left[\frac{\pi}{2}\right]$ and $z = \operatorname{Log}\frac{1}{\sqrt{3}} - i\frac{\pi}{2}$.

Residue at simple poles: $\left.\frac{1}{\cosh z - 2e^z}\right|_{z = \operatorname{Log}\frac{1}{\sqrt{3}} \pm i\frac{\pi}{2}}$

$= \left.\frac{1}{\underbrace{\sinh z + e^{-z}}_{\cosh z} - 2e^z}\right|_{z = \operatorname{Log}\frac{1}{\sqrt{3}} \pm i\frac{\pi}{2}} = e^z, \; z = \operatorname{Log}\frac{1}{\sqrt{3}} \pm \frac{i\pi}{2}$

ans: $2\pi i\left[\left. e^z\right|_{z = \operatorname{Log}\frac{1}{\sqrt{3}} + i\frac{\pi}{2}} + \left. e^z\right|_{z = \operatorname{Log}\frac{1}{\sqrt{3}} - \frac{i\pi}{2}}\right]$

$= 2\pi i\left[e^{\operatorname{Log}\frac{1}{\sqrt{3}}}(i) + e^{\operatorname{Log}\frac{1}{\sqrt{3}}}(-i)\right] = \boxed{0}$

33) For pole $\sin(z^{1/2}) = 0$, $z^{1/2} = k\pi$

$z = k^2\pi^2$, $k = 0, \pm 1, \pm 2, \ldots$ If $k = 1$, $z \approx 10$

pole is enclosed. Other poles not enclosed.

Residue is $\frac{1}{(\cos z^{1/2})\frac{1}{2}z^{-1/2}} = \left.\frac{2z^{1/2}}{\cos(z^{1/2})}\right|_{\pi^2}$

$= \frac{2\pi}{\cos(\pi)} = -2\pi$. Ans $(2\pi i)(-2\pi) = \boxed{-4\pi^2 i}$

34) $\oint \frac{dz}{\bar{z} - b} = \oint \frac{z\,dz}{|z|^2 - bz} = \oint \frac{z\,dz}{a^2 - bz}$ can do by res.

pole at $\frac{a^2}{b}$. If $a > |b|$ pole at

$a\frac{a}{b}$ is outside $|z| = a$. $\therefore$ ans $= 0$

contd next pg.

18| section 6.9

$$\frac{v(t)}{r} + C\frac{dv}{dt} = i(t), \quad \frac{V(\omega)}{r} + i\omega C V = I(\omega)$$

$$\frac{V(\omega)}{I(\omega)} = \frac{1}{\frac{1}{r} + i\omega c} = \boxed{\frac{r}{1 + i r\omega c} = \text{system func}}$$

$$V(\omega) = \left(\frac{r}{1 + i r \omega c}\right) I(\omega), \quad V(\omega) = \frac{r}{1 + i r\omega c}\frac{1}{2\pi i\omega}\left[1 - e^{-i\omega T}\right]$$

$$v(t) = \int_{-\infty}^{+\infty} \frac{r}{1 + i r\omega c}\frac{1}{2\pi i\omega}\left[1 - e^{-i\omega T}\right] e^{i\omega t}\, d\omega$$

First assume $t \geq T$. Use contour that closes in uhp. Note that integrand has removable sing. at $\omega = 0$.

Thus $v(t) = 2\pi i \operatorname{Res} \frac{r}{[1 + i(r\omega c)]}\frac{1}{2\pi i\omega}\left[1 - e^{-i\omega T}\right] e^{i\omega t}$ at $\omega = \frac{i}{cr}$

prob 18

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Now $1+2+4 \cdots\cdots 2^{n-1} = \frac{1-2^n}{1-2} = 2^n - 1$

Thus $e^{i\theta} e^{i2\theta} e^{i4\theta} \cdots e^{i2^{n-1}\theta} = e^{i(2^n-1)\theta}$

$2^n e^{i(2^n-1)\theta} \cos\theta \cos(2\theta) \cos(4\theta) \cdots \cos(2^{n-1}\theta)$

$= e^{i\theta(2^n-1)} \frac{\sin(2^n\theta)}{\sin\theta}$

Divide both sides by $2^n e^{i(2^n-1)\theta}$

12(a)

Using Maclaurin series for $\mathrm{Log}(1+z)$ but replacing z with a_n we have:

$\mathrm{Log}(1+a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} \cdots \quad |a_n|<1$

$\frac{\mathrm{Log}(1+a_n)}{a_n} = 1 - \frac{a_n}{2} + \frac{a_n^2}{3} \cdots \quad 0<|a_n|<1$

$\left|\frac{\mathrm{Log}(1+a_n)}{a_n} - 1\right| = \left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} \cdots\right|$

b) $\left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} \cdots\right| \leq \frac{|a_n|}{2} + \frac{|a_n|^2}{3} + \frac{|a_n|^3}{4} \cdots$

follows from triangle inequality.

Now $\frac{|a_n|^2}{3} \leq \frac{|a_n|^2}{2}$, $\frac{|a_n|^3}{4} \leq \frac{|a_n|^3}{2}$ etc.

Thus $\left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} \cdots\right| \leq \frac{|a_n|}{2} + \frac{|a_n|^2}{2} + \frac{|a_n|^3}{2} + \cdots$

Right side of above $= \frac{1}{2}\left[|a_n| + |a_n|^2 + |a_n|^3 \cdots\right]$