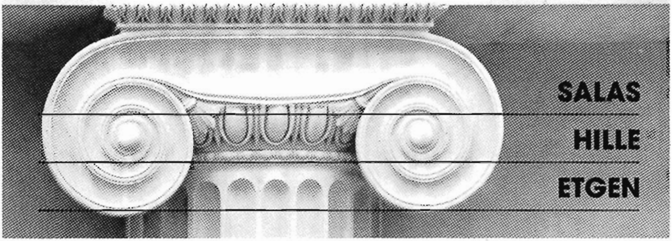
TENTH
EDITION



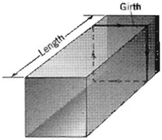
CALCULUS

ONE VARIABLE

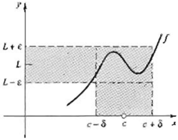
JOHN WILEY & SONS, INC.

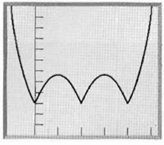
CONTENTS

CHAPTER 1 PRECALCULUS REVIEW 1



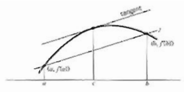
CHAPTER 2 LIMITS AND CONTINUITY 53



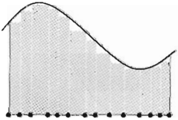


CHAPTER 3 THE DERIVATIVE; THE PROCESS OF DIFFERENTIATION 105

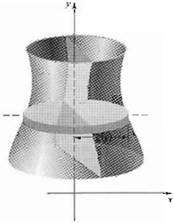
CHAPTER 4 THE MEAN-VALUE THEOREM; APPLICATIONS OF THE FIRST AND SECOND DERIVATIVES 154



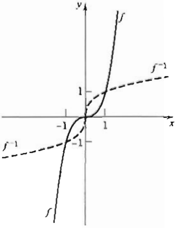
CHAPTER 5 INTEGRATION 234



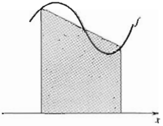
CHAPTER 6 SOME APPLICATIONS OF THE INTEGRAL 292



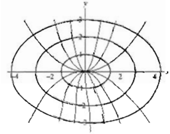
CHAPTER 7 THE TRANSCENDENTAL FUNCTIONS 333



CHAPTER 8 TECHNIQUES OF INTEGRATION 398

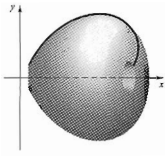


CHAPTER 9 SOME DIFFERENTIAL EQUATIONS 443

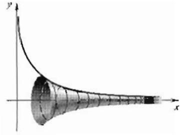


*Denotes optional section.

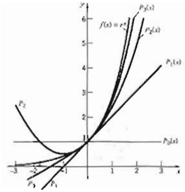
CHAPTER 10 THE CONIC SECTIONS; POLAR COORDINATES; PARAMETRIC EQUATIONS 469



CHAPTER 11 SEQUENCES; INDETERMINATE FORMS; IMPROPER INTEGRALS 528



CHAPTER 12 INFINITE SERIES 575



APPENDIX A SOME ADDITIONAL TOPICS A-1

APPENDIX B SOME ADDITIONAL PROOFS A-8

THEOREM 2.4.4

If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

The idea here is as follows: with g continuous at c, we know that

for x close to c, $g(x)$ is close to $g(c)$;

from the continuity of f at $g(c)$, we know that

with $g(x)$ close to $g(c)$, $f(g(x))$ is close to $f(g(c))$.

In summary,

with x close to c, $f(g(x))$ is close to $f(g(c))$.

The argument we just gave is too vague to be a proof. Here, in contrast, is a proof. We begin with $\epsilon > 0$. We must show that there exists a number $\delta > 0$ such that

$$\text{if} \quad |x - c| < \delta, \quad \text{then} \quad |f(g(x)) - f(g(c))| < \epsilon.$$

In the first place, we observe that, since f is continuous at $g(c)$, there does exist a number $\delta_1 > 0$ such that

$$(1) \qquad \text{if} \quad |t - g(c)| < \delta_1, \quad \text{then} \quad |f(t) - f(g(c))| < \epsilon.$$

With $\delta_1 > 0$, we know from the continuity of g at c that there exists a number $\delta > 0$ such that

$$(2) \qquad \text{if} \quad |x - c| < \delta, \quad \text{then} \quad |g(x) - g(c)| < \delta_1.$$

Combining (2) and (1), we have what we want: by (2),

$$\text{if} \quad |x - c| < \delta, \quad \text{then} \quad |g(x) - g(c)| < \delta_1$$

so that by (1)

$$|f(g(x)) - f(g(c))| < \epsilon.$$

This proof is illustrated in Figure 2.4.5. The numbers within δ of c are taken by g to within δ_1 of $g(c)$, and then by f to within ϵ of $f(g(c))$.

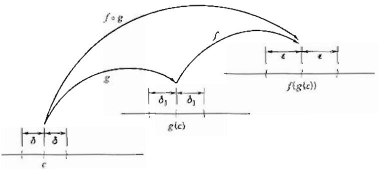


Figure 2.4.5

It's time to look at some examples.

division proposes that the road be constructed by restoring a section of the old road from A to some point P and constructing a new road from P to B. Given that the cost of restoring the old road is \$2,000,000 per mile and the cost of a new road is \$4,000,000 per mile, how much of the old road should be restored so as to minimize the cost of the project.

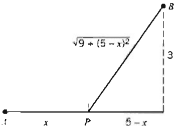


Figure 4.5.5

SOLUTION Figure 4.5.5 shows the geometry of the problem. Notice that we have chosen a straight line joining P and B rather than some curved path. (The shortest connection between two points is provided by the straight-line path.) We let x be the amount of old road that will be restored. Then

$$\sqrt{9+(5-x)^2}=\sqrt{34-10x+x^2}$$

is the length of the new part. The total cost of constructing the two sections of road is

$$C(x)=2\cdot 10^6x+4\cdot 10^6[34-10x+x^2]^{1/2},\qquad 0\le x\le 5.$$

We want to find the value of x that minimizes this function.

Differentiation gives

$$\begin{aligned}C'(x)&=2\cdot 10^6+4\cdot 10^6\left(\tfrac{1}{2}\right)[34-10x+x^2]^{-1/2}(2x-10)\\&=2\cdot 10^6+\frac{4\cdot 10^6(x-5)}{[34-10x+x^2]^{1/2}},\qquad 0<x<5.\end{aligned}$$

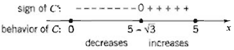
Setting $C'(x)=0$, we find that

$$\begin{aligned}1+\frac{2(x-5)}{[34-10x+x^2]^{1/2}}&=0\\2(x-5)&=-[34-10x+x^2]^{1/2}\\4(x^2-10x+25)&=34-10x+x^2\\3x^2-30x+66&=0\\x^2-10x+22&=0.\end{aligned}$$

By the general quadratic formula, we have

$$x=\frac{10\pm\sqrt{100-4(22)}}{2}=5\pm\sqrt{3}.$$

The value $x=5+\sqrt{3}$ is not in the domain of our function; the value we want is $x=5-\sqrt{3}$. We analyze the sign of C':



Since the function is continuous on $[0, 5]$, it decreases on $[0, 5-\sqrt{3}]$ and increases on $[5-\sqrt{3}, 5]$. The number $x=5-\sqrt{3}\cong 3.27$ gives the minimum value of C. The highway department will minimize its costs by restoring 3.27 miles of the old road. ❑

Example 5 (*The angle of incidence equals the angle of reflection.*) Figure 4.5.6 depicts light from point A reflected by a mirror to point B. Two angles have been marked: the *angle of incidence*, θ_i, and the *angle of reflection*, θ_r. Experiment shows

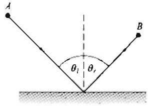


Figure 4.5.6

33. (a) Let f be continuous on $[-a, 0]$. Use a change of variable to show that

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx.$$

(b) Let f be continuous on $[-a, a]$. Show that

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]\,dx.$$

34. Let f be a function continuous on $[-a, a]$. Prove the statement basing your argument on Exercise 33.

(a) $\int_{-a}^{a} f(x)\,dx = 0 \quad$ if f is odd.

(b) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad$ if f is even.

Exercises 35–38. Evaluate using symmetry considerations.

35. $\int_{-\pi/4}^{\pi/4} (x + \sin 2x)\,dx.$ **36.** $\int_{-3}^{3} \frac{t^3}{1+t^2}\,dt.$

37. $\int_{-\pi/3}^{\pi/3} (1 + x^2 - \cos x)\,dx.$

38. $\int_{-\pi/4}^{\pi/4} (x^2 - 2x + \sin x + \cos 2x)\,dx.$

■ 5.9 MEAN-VALUE THEOREMS FOR INTEGRALS; AVERAGE VALUE OF A FUNCTION

We begin with a result that we asked you to prove earlier. (Exercise 33, Section 5.3.)

THEOREM 5.9.1 THE FIRST MEAN-VALUE THEOREM FOR INTEGRALS

If f is continuous on $[a, b]$, then there is at least one number c in (a, b) for which

$$\int_{a}^{b} f(x)\,dx = f(c)(b-a).$$

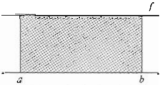
This number $f(c)$ is called *the average value* (or *mean value*) *of* f *on* $[a, b]$.

We now have the following identity:

(5.9.2) $$\boxed{\int_{a}^{b} f(x)\,dx = (\textit{the average value of } f \textit{ on } [a, b]) \cdot (b-a).}$$

This identity provides a powerful, intuitive way of viewing the definite integral. Think for a moment about area. If f is constant and positive on $[a, b]$, then Ω, the region below the graph, is a rectangle. Its area is given by the formula

$$\text{area of } \Omega = (\text{the constant value of } f \text{ on } [a, b]) \cdot (b-a). \quad \text{(Figure 5.9.1)}$$



area = (the constant value of f) $\cdot (b-a)$

Figure 5.9.1

If f is now allowed to vary continuously on $[a, b]$, then we have

$$\text{area of } \Omega = \int_{a}^{b} f(x)\,dx,$$

and the area formula reads

$$\boxed{\text{area of } \Omega = (\textit{the average value of } f \text{ on } [a, b]) \cdot (b-a).} \quad \text{(Figure 5.9.2)}$$

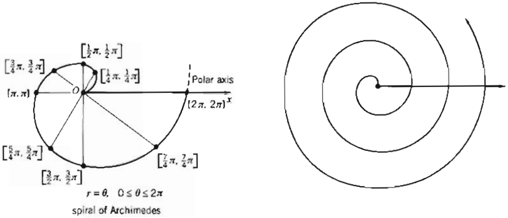


Figure 10.3.1

In Figure 10.3.2 we have marked the values of θ where r is zero and the values of θ where r takes on an extreme value.

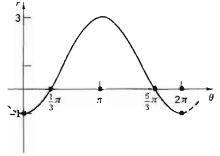


Figure 10.3.2

Reading from the figure we have the following: as θ increases from 0 to $\frac{1}{3}\pi$, r increases from -1 to 0; as θ increases from $\frac{1}{3}\pi$ to π, r increases from 0 to 3; as θ increases from π to $\frac{5}{3}\pi$, r decreases from 3 to 0; finally, as θ increases from $\frac{5}{3}\pi$ to 2π, r decreases from 0 to -1.

By applying this information step by step, we develop a sketch of the curve $r = 1 - 2\cos\theta$ in polar coordinates. (Figure 10.3.3.)

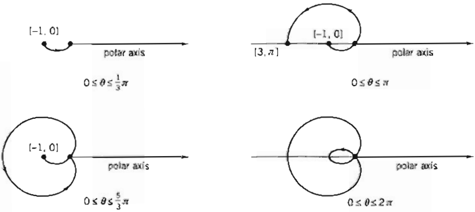
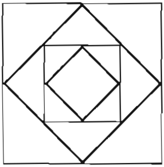


Figure 10.3.3

that remains after all of the "middle thirds" have been deleted is called the *Cantor middle third set*. Give some points that are in the Cantor set.

29. Start with a square that has sides four units long. Join the midpoints of the sides of the square to form a second square inside the first. Then join the midpoints of the sides of the second square to form a third square, and so on. See the figure. Find the sum of the areas of the squares.



30. (a) Show that if the series $\sum a_k$ converges and the series $\sum b_k$ diverges, then the series $\sum (a_k + b_k)$ diverges.

(b) Give examples to show that if $\sum a_k$ and $\sum b_k$ both diverge, then each of the series

$$\sum (a_k + b_k) \qquad \text{and} \qquad \sum (a_k - b_k)$$

may converge or may diverge.

31. Let $\sum_{k=0}^{\infty} a_k$ be a convergent series and let $R_n = \sum_{k=n+1}^{\infty} a_k$. Prove that $R_n \to 0$ as $n \to \infty$. Note that if s_n is the nth partial sum of the series, then $\sum_{k=0}^{\infty} a_k = s_n + R_n$; R_n is called the *remainder*.

32. (a) Prove that if $\sum_{k=0}^{\infty} a_k$ is a convergent series with all terms nonzero, then $\sum_{k=0}^{\infty} (1/a_k)$ diverges.

(b) Suppose that $a_k > 0$ for all k and $\sum_{k=0}^{\infty} a_k$ diverges. Show by example that $\sum_{k=0}^{\infty} (1/a_k)$ may converge and it may diverge.

33. Show that

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) \quad \text{diverges}$$

although

$$\ln\left(\frac{k+1}{k}\right) \to 0.$$

34. Show that

$$\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k \quad \text{diverges.}$$

35. (a) Assume that $d_k \to 0$ and show that

$$\sum_{k=1}^{\infty} (d_k - d_{k+1}) = d_1.$$

(b) Sum the following series:

$$\text{(i)} \sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k(k+1)}}. \qquad \text{(ii)} \sum_{k=1}^{\infty} \frac{2k+1}{2k^2(k+1)^2}.$$

36. Show that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \qquad \text{for} \qquad |x| < 1.$$

HINT: Verify that s_n, the nth partial sum of the series, satisfies the identity

$$(1-x)^2 s_n = 1 - (n+1)x^n + nx^{n+1}.$$

(Exercises 37–40. *Speed of convergence*) Find the least integer N for which the nth partial sum of the series differs from the sum of the series by less than 0.0001.

37. $\sum_{k=0}^{\infty} \frac{1}{4^k}$.

38. $\sum_{k=0}^{\infty} (0.9)^k$.

39. $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$.

40. $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$.

41. Start with the geometric series $\sum_{k=0}^{\infty} x^k$ with $|x| < 1$ and a positive number ϵ. Determine the least positive integer N for which $|L - s_N| < \epsilon$ given that the sum of the series is L and s_N is the Nth partial sum.

42. Prove that the series $\sum_{k=1}^{\infty} (a_{k+1} - a_k)$ converges iff the a_k tend to a finite limit.

■ 12.3 THE INTEGRAL TEST; BASIC COMPARISON, LIMIT COMPARISON

Here we begin our study of *series with nonnegative terms*: $a_k \geq 0$ for all k. For such series the sequence of partial sums is nondecreasing:

$$s_{n+1} = \sum_{k=0}^{n+1} a_k = a_{n+1} + \sum_{k=0}^{n} a_k \geq \sum_{k=0}^{n} a_k = s_n.$$

11. $\dfrac{1}{x^2+2x+2}$ 13. $\dfrac{2}{x\sqrt{4x^4-1}}$ 15. $\dfrac{2x}{\sqrt{1-4x^2}}+\arcsin 2x$ 17. $\dfrac{2\arcsin x}{\sqrt{1-x^2}}$ 19. $\dfrac{x-(1+x^2)\arctan x}{x^2(1+x^2)}$

21. $\dfrac{1}{(1+4x^2)\sqrt{\arctan 2x}}$ 23. $\dfrac{1}{x[1+(\ln x)^2]}$ 25. $-\dfrac{r}{|r|\sqrt{1-r^2}}$ 27. $2x\,\mathrm{arcsec}\left(\dfrac{1}{x}\right)-\dfrac{x^2}{\sqrt{1-x^2}}$

29. $\cos[\mathrm{arcsec}(\ln x)]\cdot\dfrac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}$ 31. $\sqrt{\dfrac{c-x}{c+x}}$

33. (a) x (b) $\sqrt{1-x^2}$ (c) $\dfrac{x}{\sqrt{1-x^2}}$ (d) $\dfrac{\sqrt{1-x^2}}{x}$ (e) $\dfrac{1}{\sqrt{1-x^2}}$ (f) $\dfrac{1}{x}$ 35. $\arcsin\left(\dfrac{x+b}{a}\right)+C$

39. $\frac{1}{4}\pi$ 41. $\frac{1}{4}\pi$ 43. $\frac{1}{20}\pi$ 45. $\frac{1}{24}\pi$ 47. $\dfrac{1}{3}\mathrm{arcsec}\,4-\dfrac{\pi}{9}$ 49. $\frac{1}{6}\pi$ 51. $\arctan 2-\frac{1}{4}\pi\cong 0.322$

53. $\frac{1}{2}\arcsin x^2+C$ 55. $\frac{1}{2}\arctan x^2+C$ 57. $\frac{1}{3}\arctan\left(\frac{1}{3}\tan x\right)+C$ 59. $\frac{1}{2}(\arcsin x)^2+C$ 61. $\arcsin(\ln x)+C$

63. $\dfrac{1}{3}\pi$ 65. $2\pi-\frac{4}{3}$ 67. $4\pi(\sqrt{2}-1)$

69. $\sqrt{s^2+sk}$ feet from the point where the line of the sign intersects the road.

71. (b) $\frac{1}{2}\pi a^2$; area of semicircle of radius a 75. $\dfrac{1}{\sqrt{1-x^2}}$ is not defined for $x\geq 1$.

77. estimate $\cong 0.523$, $\sin 0.523\cong 0.499$ explanation: the integral = arcsin 0.5; therefore sin (integral) = 0.5

SECTION 7.8

1. $2x\cosh x^2$ 3. $\dfrac{a\sinh ax}{2\sqrt{\cosh ax}}$ 5. $\dfrac{1}{1-\cosh x}$ 7. $ab(\cosh bx-\sinh ax)$ 9. $\dfrac{a\cosh ax}{\sinh ax}$ 11. $2e^{2x}\cosh(e^{2x})$

13. $-e^{-x}\cosh 2x+2e^{-x}\sinh 2x$ 15. $\tanh x$ 17. $(\sinh x)^x[\ln(\sinh x)+x\coth x]$ 27. absolute max -3

31. $A=2,\ B=\frac{1}{3},\ c=3$ 33. $\dfrac{1}{a}\sinh ax+C$ 35. $\dfrac{1}{3a}\sinh^3 ax+C$ 37. $\dfrac{1}{a}\ln(\cosh ax)+C$ 39. $-\dfrac{1}{a\cosh ax}+C$

41. $\frac{1}{2}(\sinh x\cosh x+x)+C$ 43. $2\cosh\sqrt{x}+C$ 45. $\sinh 1\cong 1.175$ 47. [illegible] 49. π

51. $\pi[\ln 5+\frac{1}{4}\sinh(4\ln 5)]\cong 250.492$ 53. (a) (0.69315, 1.25)
(b) $A\cong 0.38629$

SECTION 7.9

1. $2\tanh x\,\mathrm{sech}^2 x$ 3. $\mathrm{sech}\,x\,\mathrm{csch}\,x$ 5. $\dfrac{2e^{2x}\cosh(\arctan e^{2x})}{1+e^{4x}}$ 7. $\dfrac{-x\,\mathrm{csch}^2\sqrt{x^2+1}}{\sqrt{x^2+1}}$ 9. $\dfrac{-\mathrm{sech}\,x(\tanh x+2\sinh x)}{(1+\cosh x)^2}$

15. (a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $\frac{4}{3}$ (d) $\frac{5}{4}$ (e) $\frac{3}{4}$

25. (a) absolute max $f(0)=1$
(b) points of inflection at $x=\ln(1+\sqrt{2})\cong 0.881$, $x=-\ln(1+\sqrt{2})\cong-0.881$
(c) concave up on $(-\infty,-\ln(1+\sqrt{2}))\cup(\ln(1+\sqrt{2}),\infty)$; concave down on $(-\ln(1+\sqrt{2}),\ln(1+\sqrt{2}))$

(d)