Student Solutions Manual
for Jon Rogawski, Colin Adams, and Robert Franzosa's

Single Variable

CALCULUS

EARLY TRANSCENDENTALS

FOURTH EDITION

BRIAN BRADIE

Christopher Newport University

ROGER LIPSETT

Brandeis University

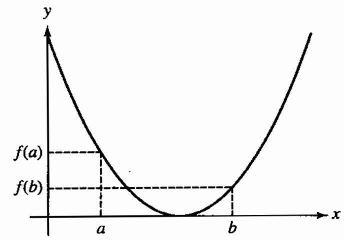
w.h.freeman
Macmillan Learning
New York

CONTENTS

4. Show that the following statement is false by drawing a graph that provides a counterexample:

If f is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

SOLUTION



5. Assume that f is continuous on $[1,5]$ and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.

(a) $f(c) = 3$ has a solution with $c \in [1,5]$.

(b) $f(c) = 75$ has a solution with $c \in [1,5]$.

(c) $f(c) = 50$ has no solution with $c \in [1,5]$.

(d) $f(c) = 30$ has exactly one solution with $c \in [1,5]$.

SOLUTION

(a) This statement is sometimes true. Because 3 does not lie between 20 and 100, the IVT cannot be used to guarantee that the function takes on the value 3 but it may still do so.

(b) This statement is always true. Because f is continuous on $[1,5]$ and $20 = f(1) < 75 < f(5) = 100$, the IVT guarantees there exists a $c \in [1,5]$ such that $f(c) = 75$.

(c) This statement is never true. Because f is continuous on $[1,5]$ and $20 = f(1) < 50 < f(5) = 100$, the IVT guarantees there exists a $c \in [1,5]$ such that $f(c) = 50$.

(d) This statement is sometimes true. Because f is continuous on $[1,5]$ and $20 = f(1) < 30 < f(5) = 100$, the IVT guarantees there exists a $c \in [1,5]$ such that $f(c) = 30$ but there may be more than one such value for c.

Exercises

1. Use the IVT to show that $f(x) = x^3 + x$ takes on the value 9 for some x in $[1,2]$.

SOLUTION Observe that $f(1) = 2$ and $f(2) = 10$. Since f is a polynomial, it is continuous everywhere; in particular on $[1,2]$. Therefore, by the IVT there is a $c \in [1,2]$ such that $f(c) = 9$.

3. Show that $g(t) = t^2 \tan t$ takes on the value $\frac{1}{2}$ for some t in $[0, \frac{\pi}{4}]$.

SOLUTION $g(0) = 0$ and $g(\frac{\pi}{4}) = \frac{\pi^2}{16}$. $g(t)$ is continuous for all t between 0 and $\frac{\pi}{4}$, and $0 < \frac{1}{2} < \frac{\pi^2}{16}$; therefore, by the IVT, there is a $c \in [0, \frac{\pi}{4}]$ such that $g(c) = \frac{1}{2}$.

5. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. *Hint:* Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.

SOLUTION Let $f(x) = x - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(1) = 1 - \cos 1 \approx .46$. Therefore, by the IVT there is a $c \in [0, 1]$ such that $f(c) = c - \cos c = 0$. Thus $c = \cos c$ and hence the equation $\cos x = x$ has a solution c in $[0, 1]$.

In Exercises 7–16, prove using the IVT.

7. $\sqrt{c} + \sqrt{c+2} = 3$ has a solution.

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{x+2} - 3$. Note that f is continuous on $[0,2]$ with $f(0) = \sqrt{0} + \sqrt{2} - 3 \approx -1.59$ and $f(2) = \sqrt{2} + \sqrt{4} - 3 \approx 0.41$. Therefore, by the IVT there is a $c \in [0,2]$ such that $f(c) = \sqrt{c} + \sqrt{c+2} - 3 = 0$. Thus $\sqrt{c} + \sqrt{c+2} = 3$, and the equation $\sqrt{c} + \sqrt{c+2} = 3$ has a solution c in $[0,2]$.

9. $\sqrt{2}$ exists. *Hint:* Consider $f(x) = x^2$.

SOLUTION Let $f(x) = x^2$. Observe that f is continuous with $f(1) = 1$ and $f(2) = 4$. Therefore, by the IVT there is a $c \in [1,2]$ such that $f(c) = c^2 = 2$. This proves the existence of $\sqrt{2}$, a number whose square is 2.

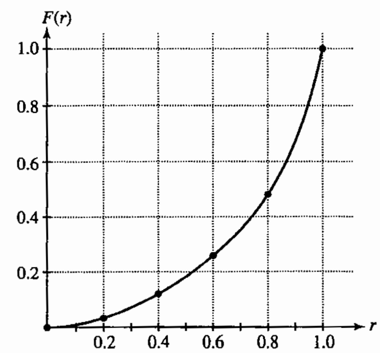
11. For all positive integers k, $\cos x = x^k$ has a solution.

SOLUTION For each positive integer k, let $f(x) = x^k - \cos x$. Observe that f is continuous on $\left[0, \frac{\pi}{2}\right]$ with $f(0) = -1$ and $f(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^k > 0$. Therefore, by the IVT there is a $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = c^k - \cos(c) = 0$. Thus $\cos c = c^k$ and hence the equation $\cos x = x^k$ has a solution c in the interval $\left[0, \frac{\pi}{2}\right]$.

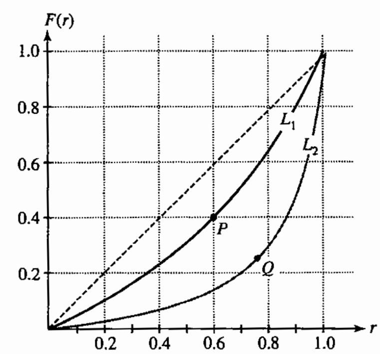
13. $2^x + 3^x = 4^x$ has a solution.

Further Insights and Challenges

*Exercises 47–49: The **Lorenz curve** $y = F(r)$ is used by economists to study income distribution in a given country (see Figure 15). By definition, $F(r)$ is the fraction of the total income that goes to the bottom rth part of the population, where $0 \le r \le 1$. For example, if $F(0.4) = 0.245$, then the bottom 40% of households receive 24.5% of the total income. Note that $F(0) = 0$ and $F(1) = 1$.*



(A) Lorenz curve for the United States in 2010



(B) Two Lorenz curves: The tangent lines at P and Q have slope 1.

FIGURE 15

47. Our goal is to find an interpretation for $F'(r)$. The average income for a group of households is the total income going to the group divided by the number of households in the group. The national average income is $A = T/N$, where N is the total number of households and T is the total income earned by the entire population.

(a) Show that the average income among households in the bottom rth part is equal to $(F(r)/r)A$.

(b) Show more generally that the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\left(\frac{F(r+\Delta r) - F(r)}{\Delta r}\right)A$$

(c) Let $0 \le r \le 1$. A household belongs to the $100r$th percentile if its income is greater than or equal to the income of $100r$ % of all households. Pass to the limit as $\Delta r \to 0$ in (b) to derive the following interpretation: A household in the $100r$th percentile has income $F'(r)A$. In particular, a household in the $100r$th percentile receives more than the national average if $F'(r) > 1$ and less if $F'(r) < 1$.

(d) For the Lorenz curves L_1 and L_2 in Figure 15(B), what percentage of households have above-average income?

SOLUTION

(a) The total income among households in the bottom rth part is $F(r)T$ and there are rN households in this part of the population. Thus, the average income among households in the bottom rth part is equal to

$$\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r}A$$

(b) Consider the interval $[r, r + \Delta r]$. The total income among households between the bottom rth part and the bottom $r + \Delta r$th part is $F(r + \Delta r)T - F(r)T$. Moreover, the number of households covered by this interval is $(r + \Delta r)N - rN = \Delta rN$. Thus, the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\frac{F(r+\Delta r)T - F(r)T}{\Delta rN} = \frac{F(r+\Delta r) - F(r)}{\Delta r} \cdot \frac{T}{N} = \frac{F(r+\Delta r) - F(r)}{\Delta r}A$$

In Exercises 5–8, assume that the radius r of a sphere is expanding at a rate of 30 cm/min. *The volume of a sphere is* $V = \frac{4}{3}\pi r^3$ *and its surface area is* $4\pi r^2$. *Determine the given rate.*

5. Volume with respect to time when $r = 15$ cm

SOLUTION As the radius is expanding at 30 centimeters per minute, we know that $\frac{dr}{dt} = 30$ cm/min. Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

Substituting $r = 15$ and $\frac{dr}{dt} = 30$ yields

$$\frac{dV}{dt} = 4\pi(15)^2(30) = 27000\pi \text{ cm}^3/\text{min}$$

7. Surface area with respect to time when $r = 40$ cm

SOLUTION Taking the derivative of both sides of $A = 4\pi r^2$ with respect to t yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. $\frac{dr}{dt} = 30$, so

$$\frac{dA}{dt} = 8\pi(40)(30) = 9600\pi \text{ cm}^2/\text{min}$$

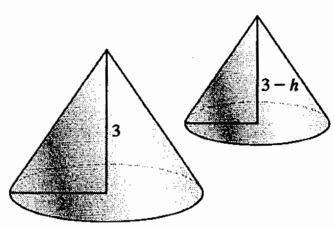
9. A conical tank (as in Example 2) has height 3 m and radius 2 m at the base. Water flows in at a rate of 2 m^3/min. How fast is the water level rising when the level is 1 m and when the level is 2 m?

SOLUTION Let V denote the volume of the water in the tank in m^3, and let h denote the height of the water in the tank in m. Now, the volume of water in the tank can be calculated as the difference between the volume of the tank and the volume of the conical space in the tank above the water. The volume of the conical tank is $\frac{1}{3}\pi(2)^2(3) = 4\pi$, and the volume of the conical space is $\frac{1}{3}\pi r^2(3-h)$, where r and $3-h$ are the base radius and the height, respectively, of the conical space. Note that the triangles highlighted in the figure below are similar; therefore, $\frac{3-h}{r} = \frac{3}{2}$. Thus, $r = \frac{2(3-h)}{3}$, so

$$V = 4\pi - \frac{1}{3}\pi\left(\frac{2(3-h)}{3}\right)^2(3-h) = 4\pi - \frac{4}{27}\pi(3-h)^3$$

Differentiating with respect to t, we obtain

$$\frac{dV}{dt} = 0 - \frac{4}{27}\pi(3(3-h)^2)\left(-\frac{dh}{dt}\right) = \frac{4}{9}\pi(3-h)^2\frac{dh}{dt}$$



Substituting 2 for $\frac{dV}{dt}$ and then solving for $\frac{dh}{dt}$ yields

$$\frac{dh}{dt} = \frac{9}{2\pi(3-h)^2}$$

When the water level is 1 m, the water level is rising at a rate of

$$\left.\frac{dh}{dt}\right|_{h=1} = \frac{9}{2\pi(2)^2} = \frac{9}{8\pi} \approx 0.3581 \text{ m/min}$$

when the water level is 2 m, the water level is rising at a rate of

$$\left.\frac{dh}{dt}\right|_{h=2} = \frac{9}{2\pi(1)^2} = \frac{9}{2\pi} \approx 1.4324 \text{ m/min}$$

In Exercises 11–14, refer to a 5-m ladder sliding down a wall, as in Figures 5 and 6. The variable h is the height of the ladder's top at time t, and x is the distance from the wall to the ladder's bottom.

11. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is 1.5 m from the wall at $t = 0$ s.

(c) From the figure, we see that

$$BQ = b - (f(x) - xf'(x)),$$
$$CQ = c - (f(x) - xf'(x))$$

and

$$PQ = \sqrt{x^2 + (f(x) - (f(x) - xf'(x)))^2} = \sqrt{x^2 + (xf'(x))^2}$$

Comparing these expressions with the numerator of $d\theta/dx$, it follows that $\frac{d\theta}{dx} = 0$ is equivalent to

$$PQ^2 = BQ \cdot CQ$$

(d) The equation $PQ^2 = BQ \cdot CQ$ is equivalent to

$$\frac{PQ}{BQ} = \frac{CQ}{PQ}$$

In other words, the sides CQ and PQ from the triangle ΔQCP are proportional in length to the sides PQ and BQ from the triangle ΔQPB. As $\angle PQB = \angle CQP$, it follows that triangles ΔQCP and ΔQPB are similar.

Seismic Prospecting *Exercises 78–80 are concerned with determining the thickness d of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point A to point D separated by a distance s. The first pulse travels directly from A to D along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to D (path ABCD), as in Figure 41. The pulse travels with velocity v_1 in the soil and v_2 in the rock.*

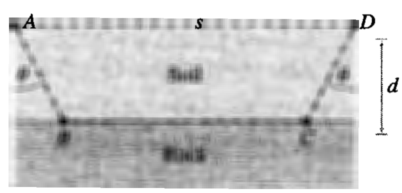


FIGURE 41

79. In this exercise, assume that $v_2/v_1 \ge \sqrt{1 + 4(d/s)^2}$.

(a) Show that inequality (1) holds if $\sin\theta = v_1/v_2$.

(b) Show that the minimal time for the second pulse is

$$t_2 = \frac{2d}{v_1}(1 - k^2)^{1/2} + \frac{s}{v_2}$$

where $k = v_1/v_2$.

(c) Conclude that $\frac{t_2}{t_1} = \frac{2d(1 - k^2)^{1/2}}{s} + k$.

SOLUTION

(a) If $\sin\theta = \frac{v_1}{v_2}$, then

$$\tan\theta = \frac{v_1}{\sqrt{v_2^2 - v_1^2}} = \frac{1}{\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1}}$$

Because $\frac{v_2}{v_1} \ge \sqrt{1 + 4(\frac{d}{s})^2}$, it follows that

$$\sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1} \ge \sqrt{1 + 4\left(\frac{d}{s}\right)^2 - 1} = \frac{2d}{s}$$

Hence, $\tan\theta \le \frac{s}{2d}$ as required.

SOLUTION

(a) Completing the square, we get

$$x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4$$

(b) Let $u = x - 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2 - 4x + 8}} = \int \frac{dx}{\sqrt{(x-2)^2 + 4}} = \int \frac{du}{\sqrt{u^2 + 4}}$$

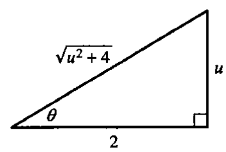
Now let $u = 2\tan\theta$. Then $du = 2\sec^2\theta\, d\theta$,

$$u^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

and

$$I = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Since $u = 2\tan\theta$, we construct a right triangle with $\tan\theta = \frac{u}{2}$:



From this we see that $\sec\theta = \sqrt{u^2+4}/2$. Thus

$$I = \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C_1 = \ln\left|\sqrt{u^2+4} + u\right| + \left(\ln\frac{1}{2} + C_1\right) = \ln\left|\sqrt{u^2+4} + u\right| + C$$

(c) Substitute back for x in the result of part (b):

$$I = \ln\left|\sqrt{(x-2)^2 + 4} + x - 2\right| + C$$

In Exercises 39–44, evaluate the integral by completing the square and using trigonometric substitution.

39. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x + 13}}$

SOLUTION First complete the square:

$$x^2 + 4x + 13 = x^2 + 4x + 4 + 9 = (x+2)^2 + 9$$

Let $u = x + 2$. Then $du = dx$, and

$$I = \int \frac{dx}{\sqrt{x^2 + 4x + 13}} = \int \frac{dx}{\sqrt{(x+2)^2 + 9}} = \int \frac{du}{\sqrt{u^2 + 9}}$$

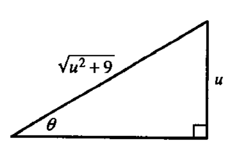
Now let $u = 3\tan\theta$. Then $du = 3\sec^2\theta\, d\theta$,

$$u^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta$$

and

$$I = \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Since $u = 3\tan\theta$, we construct the following right triangle:



(b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{2}3^{n+1} - \frac{1}{3}2^{n+1}}{\frac{1}{2}3^n - \frac{1}{3}2^n} = \lim_{n\to\infty} \frac{3^{n+2} - 2^{n+2}}{3^{n+1} - 2^{n+1}} = \lim_{n\to\infty} \frac{3 - 2\left(\frac{2}{3}\right)^{n+1}}{1 - \left(\frac{2}{3}\right)^{n+1}} = \frac{3-0}{1-0} = 3$

27. Calculate the partial sums S_4 and S_7 of the series $\sum_{n=1}^{\infty} \frac{n-2}{n^2+2n}$

SOLUTION

$$S_4 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} = -\frac{11}{60} = -0.183333$$
$$S_7 = -\frac{1}{3} + 0 + \frac{1}{15} + \frac{2}{24} + \frac{3}{35} + \frac{4}{48} + \frac{5}{63} = \frac{287}{4410} = 0.065079$$

29. Find the sum $\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$.

SOLUTION This is a geometric series with common ratio $r = \frac{2}{3}$. Therefore,

$$\frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots = \frac{\frac{4}{9}}{1-\frac{2}{3}} = \frac{4}{3}$$

31. Use series to determine a reduced fraction that has decimal expansion $0.108108108\cdots$.

SOLUTION The decimal may be regarded as a geometric series:

$$0.108108108\ldots = \frac{108}{10^3} + \frac{108}{10^6} + \frac{108}{10^9}\cdots = \sum_{n=1}^{\infty} \frac{108}{10^{3n}}$$

The series has first term $\frac{108}{10^3}$ and ratio $\frac{1}{10^3}$, so its sum is

$$0.108108108\ldots = \frac{108/10^3}{1-1/10^3} = \frac{108}{10^3}\cdot\frac{10^3}{999} = \frac{108}{999} = \frac{4}{37}$$

33. Find the sum $\sum_{n=-1}^{\infty} \frac{2^{n+3}}{3^n}$.

SOLUTION Note

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 2\sum_{n=0}^{\infty} \frac{2^n}{3^n} = 2\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

therefore,

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 2\frac{1}{1-\frac{2}{3}} = 2\cdot 3 = 6$$

35. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty}(a_n + b_n) = 1$.

SOLUTION Let $a_n = \left(\frac{1}{2}\right)^n + 1$, $b_n = -1$. The corresponding series diverge by the Divergence Test; however,

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

37. Evaluate $S = \sum_{n=3}^{\infty} \frac{1}{n(n+3)}$.

SOLUTION Note that

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

In Exercises 19–24, convert to an equation in polar coordinates of the form $r = f(\theta)$.

19. $x^2 + y^2 = 5$

SOLUTION We make the substitution $x^2 + y^2 = r^2$ to obtain; $r^2 = 5$ or $r = \sqrt{5}$.

21. $y = x^2$

SOLUTION Substituting $y = r\sin\theta$ and $x = r\cos\theta$ yields

$$r\sin\theta = r^2\cos^2\theta$$

Then, dividing by $r\cos^2\theta$ we obtain,

$$\frac{\sin\theta}{\cos^2\theta} = r \qquad \text{so} \qquad r = \tan\theta\sec\theta$$

23. $e^{\sqrt{x^2+y^2}} = 1$

SOLUTION Make the substitution $r = \sqrt{x^2+y^2}$ to get $e^r = 1$. Taking logs of both sides gives $r = 0$. The graph of this equation consists only of the origin.

25. Match each equation with its description:

(a) $r = 2$	**(i)** Vertical line
(b) $\theta = 2$	**(ii)** Horizontal line
(c) $r = 2\sec\theta$	**(iii)** Circle
(d) $r = 2\csc\theta$	**(iv)** Line through origin

SOLUTION

(a) $r = 2$ describes the points 2 units from the origin. Hence, it is the equation of a circle.

(b) $\theta = 2$ describes the points P so that $\overline{OP}$ makes an angle of $\theta_0 = 2$ with the positive x-axis. Hence, it is the equation of a line through the origin.

(c) This is $r\cos\theta = 2$, which is $x = 2$, a vertical line.

(d) Converting to rectangular coordinates, we get $r = 2\csc\theta$, so $r\sin\theta = 2$ and $y = 2$. This is the equation of a horizontal line.

27. Find the values of θ in the plot of $r = 4\cos\theta$ corresponding to points A, B, C, D in Figure 19. Then indicate the portion of the graph traced out as θ varies in the following intervals:

(a) $0 \le \theta \le \frac{\pi}{2}$ **(b)** $\frac{\pi}{2} \le \theta \le \pi$ **(c)** $\pi \le \theta \le \frac{3\pi}{2}$

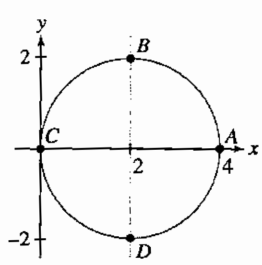


FIGURE 19 Plot of $r = 4\cos\theta$.

SOLUTION The point A is on the x-axis hence $\theta = 0$. The point B is in the first quadrant with $x = y = 2$; hence $\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$. The point C is at the origin. Thus,

$$r = 0 \Rightarrow 4\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

The point D is in the fourth quadrant with $x = 2$, $y = -2$, hence

$$\theta = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$0 \le \theta \le \frac{\pi}{2}$ represents the first quadrant; hence the points (r, θ) where $r = 4\cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$ are the points on the circle which are in the first quadrant, as shown below: