Applied Quantum Mechanics

Chapter 1 Problems and Solutions

LAST NAME FIRST NAME

Useful constants MKS (SI)

Speed of light in free space	$c = 2.99792458 \times 10^{8}$ m s^{-1}
Planck's constant	$\hbar = 6.58211889(26) \times 10^{-16}$ eV s
	$\hbar = 1.054571596(82) \times 10^{-34}$ J s
Electron charge	$e = 1.602176462(63) \times 10^{-19}$C
Electron mass	$m_0 = 9.10938188(72) \times 10^{-31}$ kg
Neutron mass	$m_n = 1.67492716(13) \times 10^{-27}$ kg
Proton mass	$m_p = 1.67262158(13) \times 10^{-27}$ kg
Boltzmann constant	$k_B = 1.3806503(24) \times 10^{-23}$ J K^{-1}
	$k_B = 8.617342(15) \times 10^{-5}$ eV K^{-1}
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}$ F m^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0\mu_0}$
Avagadro's number	$N_A = 6.02214199(79) \times 10^{23}$ mol^{-1}
Bohr radius	$a_B = 0.52917721(19)\times10^{-10}$m

$$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_0e^2}$$

Inverse fine-structure constant $\alpha^{-1} = 137.0359976(50)$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

PROBLEM 1

A metal ball is buried in an ice cube that is in a bucket of water.

(a) If the ice cube with the metal ball is initially under water, what happens to the water level when the ice melts?

(b) If the ice cube with the metal ball is initially floating in the water, what happens to the water level when the ice melts?

(c) Explain how the Earth's average sea level could have increased by at least 100 m compared to about 20,000 years ago.

(d) Estimate the thickness and weight per unit area of the ice that melted in (c). You may wish to use the fact that the density of ice is 920 kg m^{-3}, today the land surface area of the Earth is about 148,300,000 km^2 and water area is about 361,800,000 km^2.

PROBLEM 2

Sketch and find the volume of the largest and smallest convex plug manufactured from a sphere of radius r = 1 cm to fit exactly into a circular hole of radius r = 1 cm, an isosceles triangle with base 2 cm and a height h = 1 cm, and a half circle radius r = 1 cm and base 2 cm.

PROBLEM 3

An initially stationary particle mass m_1 is on a frictionless table surface and another particle mass m_2 is positioned vertically below the edge of the table. The distance from the particle mass m_1 to the edge of the table is l. The two particles are connected by a taught, light, inextensible string of length $L > l$.

(a) How much time elapses before the particle mass m_1 is launched off the edge of the table?

(b) What is the subsequent motion of the particles?

(c) How is your answer for (a) and (b) modified if the string has spring constant κ?

PROBLEM 4

The velocity of water waves in shallow water may be approximated as $v = \sqrt{gh}$ where g is the acceleration due to gravity and h is the depth of the water. Sketch the lowest frequency standing water wave in a 5 m long garden pond that is 0.9 m deep and estimate its frequency.

PROBLEM 5

(a) What is the dispersion relation of a wave whose group velocity is half the phase velocity?

(b) What is the dispersion relation of a wave whose group velocity is twice the phase velocity?

(c) What is the dispersion relation when the group velocity is four times the phase velocity?

PROBLEM 6

A stationary ground-based radar uses a continuous electromagnetic wave at 10 GHz frequency to measure the speed of a passing airplane moving at a constant altitude and in a straight line at 1000 km hr^{-1}. What is the maximum beat frequency between the out going and reflected radar beams? Sketch how the beat frequency varies as a function of time. What happens to the beat frequency if the airplane moves in an arc?

PROBLEM 7

How would Maxwell's equations be modified if magnetic charge g (magnetic monopoles) were discovered? Derive an expression for conservation of magnetic current and write down a generalized Lorentz force law that includes magnetic charge. Write Maxwell's equations with magnetic charge in terms of a field $\mathbf{G} = \sqrt{\varepsilon}\mathbf{E} + i\sqrt{\mu}\mathbf{H}$.

PROBLEM 8

The capacitance of a small metal sphere in air is $C_0 = 1.1 \times 10^{-18}\ F$. A thin dielectric film with relative permittivity $\varepsilon_{r_1} = 10$ uniformly coats the sphere and the capacitance increases to $2.2 \times 10^{-18}\ F$. What is the thickness of the dielectric film and what is the single electron charging energy of the dielectric coated metal sphere?

PROBLEM 9

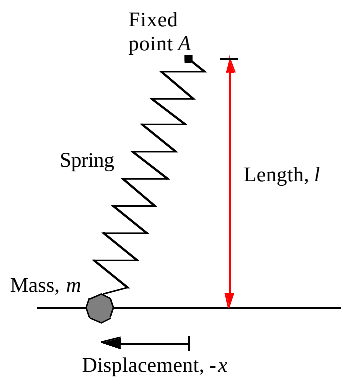
(a) A diatomic molecule has atoms with mass m_1 and m_2. An isotopic form of the molecule has atoms with mass m'_1 and m'_2. Find the ratio of vibration oscillation frequency ω / ω' of the two molecules.

(b) What is the ratio of vibrational frequencies for carbon monoxide isotope 12 ($^{12}C^{16}O$) and carbon monoxide isotope 13 ($^{13}C^{16}O$)?

PROBLEM 10

(a) Find the frequency of oscillation of the particle of mass m illustrated in the Fig. The particle is only free to move along a line and is attached to a light spring whose other end is fixed at point A located a distance l perpendicular to the line. A force F_0 is required to extend the spring to length l.

(b) Part (a) describes a new type of child's swing. If the child weighs 20 kg, the length $l = 2.5$ m, and the force $F_0 = 450$ N, what is the period of oscillation?



PROBLEM 1

Prove that particle flux (current) is zero if the one-dimensional exponential decaying wave function in tunnel barrier of energy V_0 and finite thickness L is $\psi(x, t) = Be^{-\kappa x - i\omega t}$, where κ is a real positive number and particle energy $E = \hbar\omega < V_0$.

PROBLEM 2

(a) Use a Taylor expansion to show that the second derivative of a wavefunction $\psi(x)$ sampled at positions $x_j = jh_0$, where j is an integer and h_0 is a small fixed increment in distance x, may be approximated as

$$\frac{d^2}{dx^2}\psi(x_j) = \frac{\psi(x_{j-1}) - 2\psi(x_j) + \psi(x_{j+1})}{h_0^2}$$

(b) By keeping additional terms in the expansion, show that a more accurate approximation of the second derivative is

$$\frac{d^2}{dx^2}\psi(x_j) = \frac{-\psi(x_{j-2}) + 16\psi(x_{j-1}) - 30\psi(x_j) + 16\psi(x_{j+1}) - \psi(x_{j+2})}{12h_0^2}$$

PROBLEM 3

Using the method outlined in Exercise 7 of Chapter 3 as a starting point, calculate numerically the first four energy eigenvalues and eigenfunctions for an electron with effective mass $m_e^* = 0.07 \times m_0$ confined to a potential well $V(x) = V_0$ of width L = 10 nm with *periodic* boundary conditions.

Periodic boundary conditions require that the wave function at position $x = 0$ is connected (wrapped around) to position $x = L$. The wave function and its first derivative are continuous and smooth at this connection.

Your solution should include plots of the eigenfunctions and a listing of the computer program you used to calculate the eigenfunctions and eigenvalues.

PROBLEM 4

Using the method outlined in Exercise 7 of Chapter 3 as a starting point, calculate numerically the first six energy eigenvalues and eigenfunctions for an electron with effective mass $m_e^* = 0.07 \times m_0$ confined to a triangular potential well of width L = 20 nm bounded by barriers of infinite energy at $x < 0$ and $x > L$. The triangular potential well as a function of distance x is given by $V(x) = V_0 \times x / L$ where V_0 = 1 eV.

Explain the change in shape of the wave function with increasing eigenenergy.

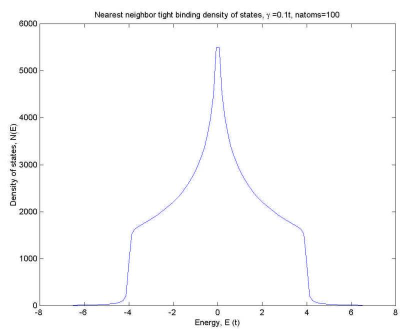
Your solution should include plots of the eigenfunctions and a listing of the computer program you used to calculate the eigenfunctions and eigenvalues.

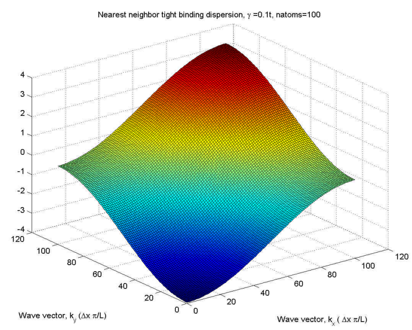
PROBLEM 5

Calculate the transmission and reflection coefficient for an electron of energy E, moving from left to right, impinging normal to the plane of a semiconductor heterojunction potential barrier of energy V_0, where the effective electron mass on the left-hand side is m_1 and the effective electron mass on the right-hand side is m_2.

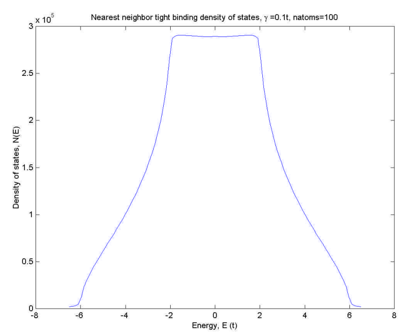
If the potential barrier energy is V_0 = 1.5 eV and the ratio of effective electron mass on either side of the heterointerface is m_1 / m_2 = 3, at what particle energy is the transmission coefficient unity? What is the transmission coefficient in the limit $E \to \infty$?

(c) For the square lattice with nearest neighbor interactions only, we have



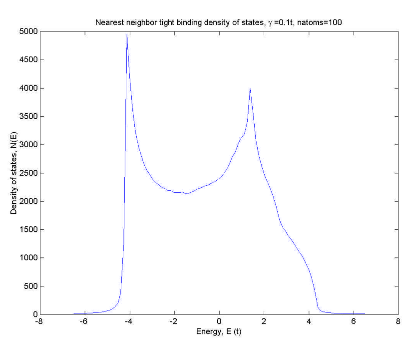


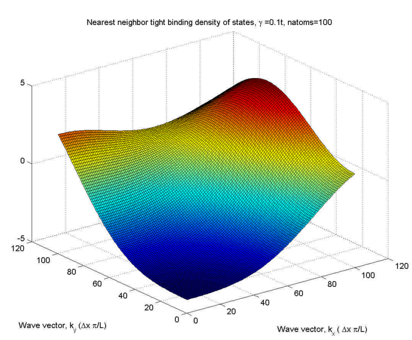
and for the cubic lattice



Notice that the band edge energy for simple cubic lattice of dimension d occurs at $\pm 2td$.

(d) For the hexagonal lattice with second nearest neighbor interactions





which is different compared to the result for the hexagonal lattice with only nearest neighbor interactions shown in the following figure.

(b) To find the value of the product in uncertainty in position Δx and momentum Δp_x for the first exited state of a particle of mass m in a one-dimensional harmonic oscillator potential we use

$\Delta x = (\langle x^2\rangle - \langle x\rangle^2)^{1/2}$

and

$\Delta p_x = (\langle p^2{}_x\rangle - \langle p_x\rangle^2)^{1/2}$

and since $\langle x\rangle = 0$ and $\langle p_x\rangle = 0$ we will be interested in finding the value of $\langle x^2\rangle$ and $\langle p^2{}_x\rangle$. Starting with $\langle x^2\rangle$, we have

$$\langle \hat{x}^2\rangle = \left(\frac{\hbar}{2m\omega}\right)\langle 1|(\hat{b}+\hat{b}^\dagger)^2|1\rangle = \left(\frac{\hbar}{2m\omega}\right)\langle 1|\hat{b}\hat{b}+\hat{b}^\dagger\hat{b}^\dagger+\hat{b}\hat{b}^\dagger+\hat{b}^\dagger\hat{b}|1\rangle = \frac{3\hbar}{2m\omega}$$

and one can see that for the general state $|n\rangle$ one has $\langle x^2\rangle = \left(\frac{\hbar}{2m\omega}\right)(1+2n)$. Now turning our attention to $\langle p^2{}_x\rangle$ we have

$$\langle p^2{}_x\rangle = \left(\frac{\hbar m\omega}{2}\right)\langle 1|(\hat{b}^\dagger-\hat{b})^2|1\rangle = \left(\frac{\hbar m\omega}{2}\right)\langle 1|-\hat{b}\hat{b}-\hat{b}^\dagger\hat{b}^\dagger+\hat{b}\hat{b}^\dagger+\hat{b}^\dagger\hat{b}|1\rangle = \frac{3\hbar m\omega}{2}$$

and one can see that for the general state $|n\rangle$ one has $\langle \hat{p}_x^2\rangle = \left(\frac{\hbar m\omega}{2}\right)(1+2n)$.

For the particular case we are interested in $|n = 1\rangle$ and the uncertainty product is

$$\Delta x\Delta p_x = (\langle x^2\rangle\langle p^2{}_x\rangle)^{1/2} = \frac{3}{2}\hbar$$

For the general state $|n\rangle$ the uncertainty product $\Delta x\Delta p_x = \frac{\hbar}{2}(1+2n)$

Solution 3

(a) We start with the reasonable assumption that the expectation value of an observable associated with operator $\hat{\mathbf{A}}$ evolves smoothly in time such that

$$\left|\frac{d}{dt}\langle\hat{\mathbf{A}}\rangle\right| = \frac{\Delta A}{\Delta t}$$

which may be written as

$$\Delta t = \frac{\Delta A}{\left|\frac{d}{dt}\langle\hat{\mathbf{A}}\rangle\right|}$$

In this problem the operator $\hat{\mathbf{A}}$ is time-independent so that

$$\frac{d}{dt}\langle\hat{\mathbf{A}}\rangle = \frac{i}{\hbar}\langle[H,\hat{\mathbf{A}}]\rangle + \langle\frac{\partial}{\partial t}\hat{\mathbf{A}}\rangle = \frac{i}{\hbar}\langle[H,\hat{\mathbf{A}}]\rangle$$

since $\langle\frac{\partial}{\partial t}\hat{\mathbf{A}}\rangle = 0$

In addition, the generalized uncertainty relation for operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ is

$$\Delta A\Delta B \geq \frac{i}{2}[\langle\hat{\mathbf{A}},\hat{\mathbf{B}}\rangle]$$

which may be re-written as

$$2\Delta A\Delta B \geq |\langle[\hat{\mathbf{A}},\hat{\mathbf{B}}]\rangle|$$

Applied Quantum Mechanics

Final

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SI-MKS

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