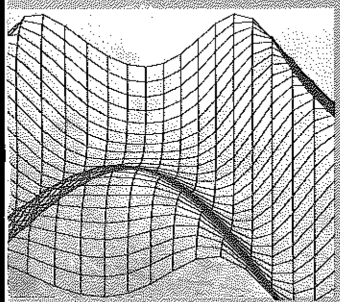

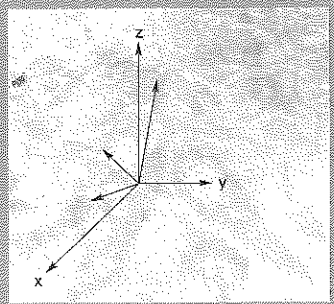

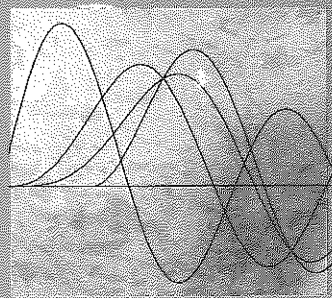

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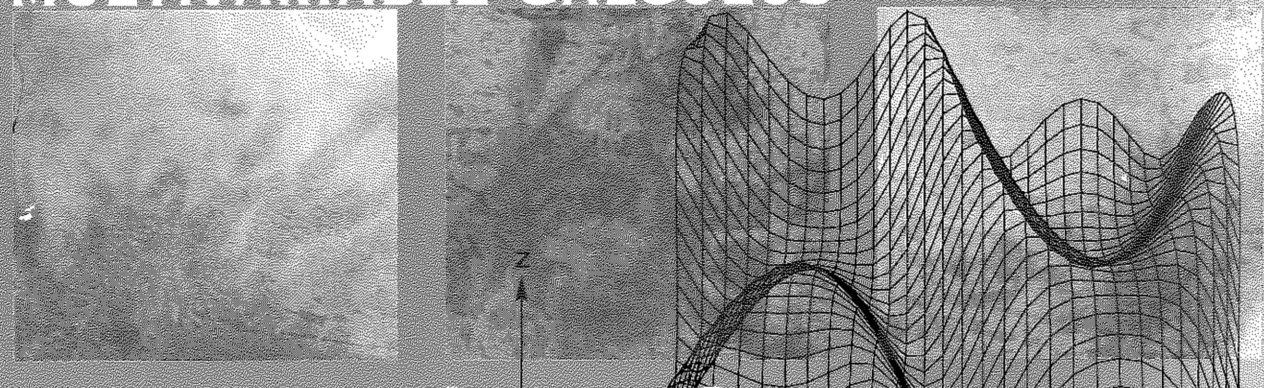

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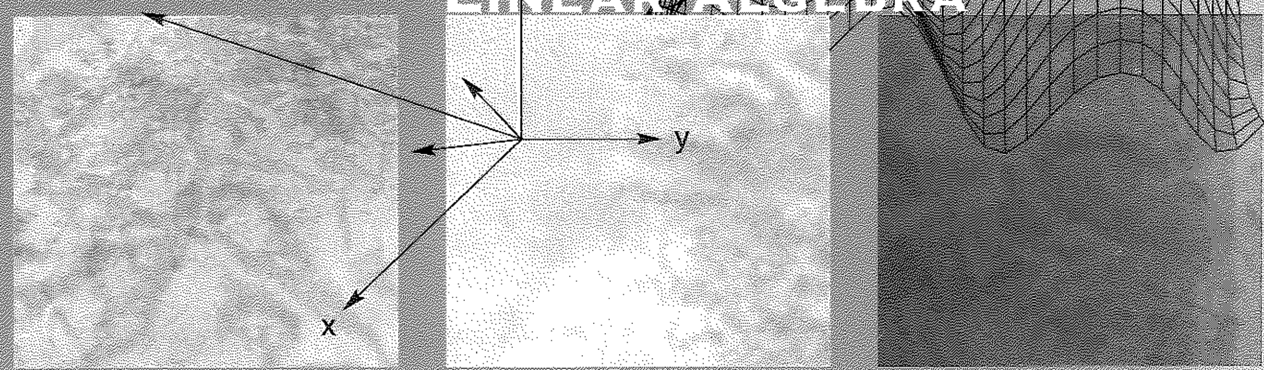

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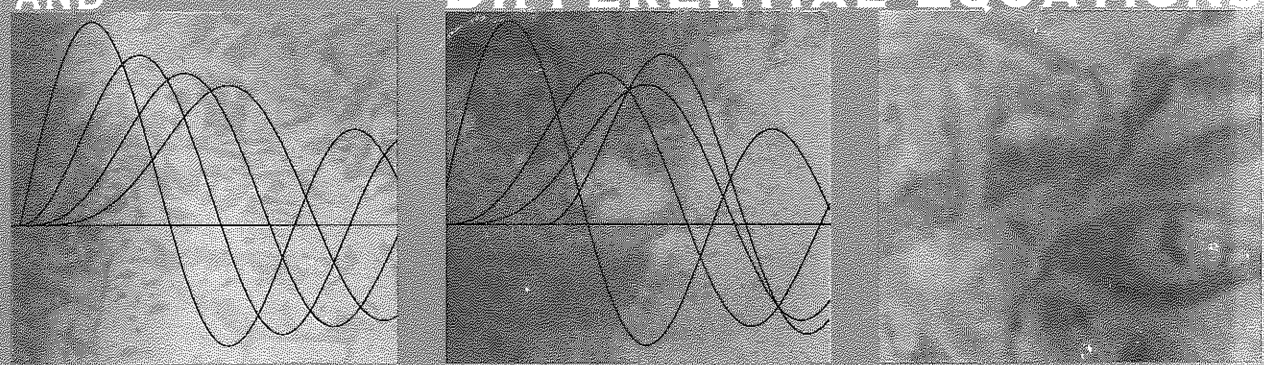

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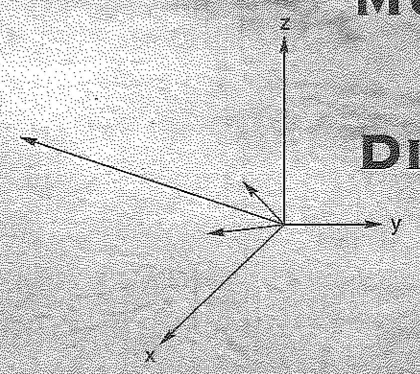

MULTIVARIABLE CALCULUS LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

THIRD EDITION

STANLEY I. GROSSMAN

Leon Gerber
St. John's University

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F I V E

INTRODUCTION TO VECTOR ANALYSIS

5.1 VECTOR FIELDS

Vector Field $\mathbf{F}(\mathbf{x})$, a vector-valued function of a vector. If $\mathbf{F} = \nabla f$ for some function f, $\mathbf{F}$ is *conservative* and f is a *potential* for $\mathbf{F}$.

Conservation of Energy $\frac{1}{2}m|\mathbf{x}'|^2 + f(\mathbf{x}) =$ kinetic + potential energy = C.

Problems 5.1

In Problems 1–21, compute the gradient of the function.

1. $\nabla(x^2+y^2)^{-1/2} = -\frac{1}{2}(x^2+y^2)^{-3/2}\nabla(x^2+y^2) = -(x^2+y^2)^{-3/2}(x\mathbf{i}+y\mathbf{j})$

3. $\nabla(x+y)^2 = 2(x+y)\nabla(x+y) = 2(x+y)(\mathbf{i}+\mathbf{j})$

5. $\nabla\cos(x-y) = -\sin(x-y)\nabla(x-y) = -\sin(x-y)(\mathbf{i}-\mathbf{j})$

7. $\nabla[y\tan(y-x)] = -y\sec^2(y-x)\mathbf{i} + [\tan(y-x) + y\sec^2(y-x)]\mathbf{j}$

9. $\nabla\sec(x+3y) = \sec(x+3y)\tan(x+3y)\nabla(x+3y) = \sec(x+3y)\tan(x+3y)(\mathbf{i}+3\mathbf{j})$

11. $f = \frac{x^2-y^2}{x^2+y^2} = 1 - \frac{2y^2}{x^2+y^2} = \frac{2x^2}{x^2+y^2} - 1$. $\nabla f = [\frac{2y^2}{(x^2+y^2)^2}\cdot 2x]\mathbf{i} + [\frac{-2x^2}{(x^2+y^2)^2}\cdot 2y]\mathbf{j} = \frac{4xy}{(x^2+y^2)^2}[y\mathbf{i} - x\mathbf{j}]$

13. $\nabla(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\nabla(x^2+y^2+z^2) = (x^2+y^2+z^2)^{-1/2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

15. $\nabla\sin x\cos y\tan z = \cos x\cos y\tan z\mathbf{i} - \sin x\sin y\tan z\mathbf{j} + \sin x\cos y\sec^2 z\mathbf{k}$

17. $\nabla(x\ln y - z\ln x) = (\ln y - z/x)\mathbf{i} + (x/y)\mathbf{j} - \ln x\mathbf{k}$

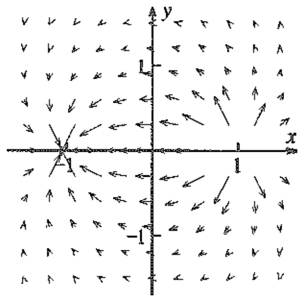
19. $\nabla[(y-z)e^{x+2y+3z}] = e^{x+2y+3z}[(y-z)\mathbf{i} + (1+2y-2z)\mathbf{j} + (-1+3y-3z)\mathbf{k}]$

21. $\nabla\ln\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = \frac{1}{2}\nabla\{\ln[(x-1)^2+y^2] - \ln[(x+1)^2+y^2]\}$

$= [\frac{x-1}{(x-1)^2+y^2} - \frac{x+1}{(x+1)^2+y^2}]\mathbf{i} + [\frac{y}{(x-1)^2+y^2} - \frac{y}{(x+1)^2+y^2}]\mathbf{j}$

$= \frac{[(x-1)(x^2+y^2+1+2x) - (x+1)(x^2+y^2+1-2x)]}{[(x-1)^2+y^2][(x+1)^2+y^2]}\mathbf{i} +$

$y\frac{x^2+y^2+1+2x-(x^2+y^2+1-2x)}{[(x-1)^2+y^2][(x+1)^2+y^2]}\mathbf{j} = \frac{2(x^2-y^2-1)\mathbf{i} + 4xy\mathbf{j}}{[(x-1)^2+y^2][(x+1)^2+y^2]}$



The figure shows $\frac{1}{8}\nabla f$, which is unbounded near $(\pm 1, 0)$.

23. Show that $y\mathbf{i} + x\mathbf{j}$ is conservative. ▹ $-\nabla(-xy) = y\mathbf{i} + x\mathbf{j}$

25. Show that $-\alpha\mathbf{x}/|\mathbf{x}|^k$ is conservative.
▹ If $k = 2$, then $-\nabla[\frac{1}{2}\alpha\ln(x^2+y^2+z^2)] = -\alpha(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/(x^2+y^2+z^2) = -\alpha\mathbf{x}/|\mathbf{x}|^2$. Otherwise
$-\nabla[\alpha(x^2+y^2+z^2)^{1-k/2}] = [-2(1-\frac{1}{2}k)/(2-k)]\alpha(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})(x^2+y^2+z^2)^{-k/2} = -\alpha\mathbf{x}/|\mathbf{x}|^k$

27. Show that $y\mathbf{i} - x\mathbf{j}$ is not conservative. ▹ $-x = f_y \Rightarrow f = -xy + g(y) \Rightarrow f_x = -y$ which contradicts $f_x = y$.

5.2 WORK AND LINE INTEGRALS

Piecewise Smooth Curve C: join a finite number of smooth curves end to end.

Work = Line Integral $W = \int_C \mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)dt = \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t)y'(t)]dt$
All parametrizations of the path yield the same result. See Problem 36.

If $\mathbf{F} = P(x)\mathbf{i} + Q(y)\mathbf{j}$ then $\int_C \mathbf{F}(\mathbf{x})\cdot d\mathbf{x} = \int P(x)dx + \int Q(y)dy$.

Problems 5.2

In Problems 1–6, find the work W joules when the force F newtons in direction θ moves an object $\overline{PQ}$ meters.

1. $F = 3$, $\theta = 0$, $P = (2,3)$, $Q = (1,7)$ ▹ $\mathbf{F} = (3,0)$, $\mathbf{d} = (1,7) - (2,3) = (-1,4)$. $W = \mathbf{F}\cdot\mathbf{d} = -3$

3. $F = 6$, $\theta = \frac{1}{4}\pi$, $P = (2,3)$, $Q = (-1,4)$
▹ $\mathbf{F} = (6\cos\frac{1}{4}\pi, 6\sin\frac{1}{4}\pi) = (3\sqrt{2}, 3\sqrt{2})$. $\mathbf{d} = (-1,4) - (2,3) = (-3,1)$. $W = -6\sqrt{2}$

5. $F = 4$, θ has direction $2\mathbf{i} + 3\mathbf{j}$, $P = (2,0)$, $Q = (-1,3)$
▹ $\mathbf{F} = 4(2,3)/\sqrt{2^2+3^2} = (8,12)/\sqrt{13}$. $\mathbf{d} = (-1,3) - (2,0) = (-3,3)$. $W = 12/\sqrt{13}$

In Problems 7–34, calculate $W = \int_C \mathbf{F}(\mathbf{x})\cdot d\mathbf{x}$. In Problems 27–43, W is the number of newtons.

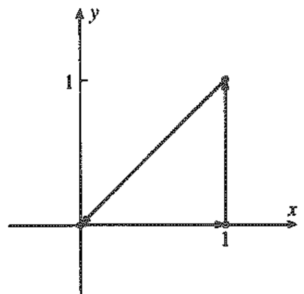
7. $\mathbf{F} = (xy, ye^x)$, $x = 2-t$, $y = 1$, $0 \le t \le 2$. $W = \int_0^2 (2-t, e^{2-t})\cdot(-1,0)dt = \int_0^2 (t-2)dt = \frac{(t-2)^2}{2}\Big|_0^2 = -2$

9. $\mathbf{F} = (x^2, y^2)$; C is segment $(0,0)$ to $(2,4) \Rightarrow x = 2t$, $y = 4t$, $0 \le t \le 1$.
$W = \int_0^1 [(2t)^2, (4t)^2]\cdot[2,4]dt = \int(4t^2, 16t^2)\cdot(2,4)dt = \int(4t^2\cdot 2 + 16t^2\cdot 4)dt = \int 72t^2dt = 24t^3\Big|_0^1 = 24$

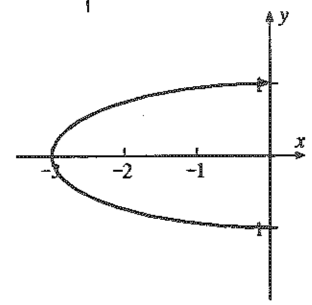
11. $\mathbf{F} = (xy, y-x)$; C is segment $y = 2x-4$ from $(1,-2)$ to $(2,0) \Rightarrow x = t$, $y = 2t-4$, $1 \le t \le 2$.
$W = \int_0^2 [t(2t-4), 2t-4-t]\cdot[1,2]dt = \int(2t^2-4t, t-4)\cdot(1,2)dt = \int[(2t^2-4t)1 + (t-4)2]dt$
$= \int(2t^2-2t-8)dt = \left[\frac{2}{3}t^3 - t^2 - 8t\right]_1^2 = \frac{2}{3}(8-1) - (4-1) - 8(2-1) = -\frac{19}{3}$

13. $\mathbf{F} = (xy, y-x)$; C is unit circle counterclockwise $\Rightarrow x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.
$W = \int_0^{2\pi}(\cos t\sin t, \sin t - \cos t)\cdot(-\sin t, \cos t)dt = \int[\cos t\sin t(-\sin t) + (\sin t - \cos t)\cos t]dt$
$= \int(-\cos t\sin^2 t + \sin t\cos t - \cos^2 t)dt = \left[-\frac{1}{3}\sin^3 t + \frac{1}{2}\sin^2 t - \frac{1}{2}\cos t\sin t - \frac{1}{2}t\right]_0^{2\pi} = -0 + 0 - 0 - \pi = -\pi$

15. $\mathbf{F} = (xy, y-x)$; $C_1 = (0,0)(1,0) \Rightarrow x = t$, $y = 0$, $0 \le t \le 1$; $C_2 = (1,0)(1,1)$
$\Rightarrow x = 1$, $y = t$, $0 \le t \le 1$; $C_3 = (1,1)(0,0) \Rightarrow x = t$, $y = t$, $1 \ge t \ge 0$.
$W = \int_0^1 (t\cdot 0, 0-t)\cdot(1,0)dt + \int_0^1 (1\cdot t, t-1)\cdot(0,1)dt + \int_1^0 (t\cdot t, t-t)(1,1)dt$
$= \int_0^1 (0,-t)\cdot(1,0)dt + \int_0^1 (t, t-1)\cdot(0,1)dt - \int_0^1 (t^2, 0)\cdot(1,1)dt$
$= 0 + \int_0^1 (t-1)dt - \int_0^1 t^2dt = \frac{1}{2}(t-1)^2\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1 = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$



17. $\mathbf{F} = (x^2+2y, -y^2)$; C is $x^2/9 + y^2 = 1$ clockwise from $(0,-1)$ to $(0,1)$
$\Rightarrow x = 3\cos t$, $y = \sin t$, $t = 3\pi/2$ to $\pi/2$.
$W = \int_{3\pi/2}^{\pi/2}(9\cos^2 t + 2\sin t, -\sin^2 t)\cdot(-3\sin t, \cos t)dt$
$= \int[9\cos^2 t + 2\sin t)(-3\sin t) + (-\sin^2 t)\cos t]dt$
$= \int[27\cos^2 t(-\sin t) - 6\sin^2 t - \sin^2 t\cdot\cos t]dt$
$= \left[9\cos^3 t + 3\sin t\cos t - 3t - \frac{1}{3}\sin^3 t\right]_{3\pi/2}^{\pi/2} = 0 + 0 + 3\pi - \frac{2}{3} = 3\pi - \frac{2}{3}$



19. $\mathbf{F} = (e^{x+y}, e^{x-y})$; $C_1 = (0,0)(1,0) \Rightarrow x = t$, $y = 0$, $0 \le t \le 1$; $C_2 = (1,0)(0,1) \Rightarrow x = 1-t$, $y = t$, $0 \le t \le 1$;
$C_3 = (0,1)(0,0)$, $x = 0$, $y = t$, $1 \ge t \ge 0$. $W = \int_0^1 (e^{t+0}, e^{t-0})\cdot(1,0)dt + \int_0^1 (e^{1-t+t}, e^{1-t-t})\cdot(-1,1)dt$
$+ \int_1^0 (e^{0+t}, e^{0-t})\cdot(0,1)dt = \int_0^1 e^t dt + \int_0^1 (-e + e^{1-2t})dt - \int_0^1 e^{-t}dt = \left[e^t - et - \frac{1}{2}e^{1-2t} + e^{-t}\right]_0^1$
$= (e-1) - e - \frac{1}{2}(e^{-1} - e) + (e^{-1} - 1) = \frac{1}{2}e + \frac{1}{2}e^{-1} - 2$ « $= \cosh 1 - 2$»

9. $\begin{pmatrix}-3\\4\\2\end{pmatrix}, \begin{pmatrix}7\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\1\\8\end{pmatrix}$ ▷ $\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}1&1&8\\-3&4&2\\7&-1&3\end{pmatrix} \begin{array}{c}R_1\\R_2+3R_1\\R_3-7R_1\end{array}\begin{pmatrix}1&1&8\\0&7&26\\0&-8&-53\end{pmatrix} \begin{array}{c}R_1\\R_2\\7R_3+8R_2\end{array}\begin{pmatrix}1&1&8\\0&7&26\\0&0&-163\end{pmatrix}$ ind(s)

11. $\begin{pmatrix}1\\-2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\0\\2\\-2\end{pmatrix}, \begin{pmatrix}0\\4\\-1\\1\end{pmatrix}, \begin{pmatrix}5\\0\\3\\-1\end{pmatrix}$ ∂▷ $\begin{array}{c}R_1\\R_2\\R_3\\R_4\end{array}\begin{pmatrix}1&-2&1&1\\3&0&2&-2\\0&4&-1&1\\5&0&3&-1\end{pmatrix} \begin{array}{c}R_1\\R_2-3R_1\\R_3\\R_4-5R_1\end{array}\begin{pmatrix}1&-2&1&1\\0&6&-1&-5\\0&4&-1&1\\0&10&-2&-6\end{pmatrix}$ ind (s)

13. P_2: $1-x$, x ▷ $\begin{pmatrix}1&-1\\0&1\end{pmatrix}$ independent (span)

15. P_2: $1-x$, $1+x$, x^2 ▷ $\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}1&-1&0\\1&1&0\\0&0&1\end{pmatrix} \begin{array}{c}R_1\\R_2-R_1\\R_3\end{array}\begin{pmatrix}1&-1&0\\0&2&0\\0&0&1\end{pmatrix}$ independent (s)

17. P_3: $2x$, x^3-3, $1+x-4x^3$, $x^3+18x-9$ No x^2 term. 4 vectors in $P_3 = \mathbb{R}^3$ that don't span aren't independent.

19. M_{22}: $\begin{pmatrix}1&-1\\0&6\end{pmatrix}, \begin{pmatrix}-1&0\\3&1\end{pmatrix}, \begin{pmatrix}1&1\\-1&2\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}$ ∂▷ $\begin{array}{c}R_1\\R_2\\R_3\\R_4\end{array}\begin{pmatrix}1&-1&0&6\\-1&0&3&1\\1&1&-1&2\\0&1&1&0\end{pmatrix} \begin{array}{c}R_1\\R_2+R_1\\R_3-R_1\\R_4\end{array}\begin{pmatrix}1&-1&0&6\\0&-1&3&7\\0&2&-1&-4\\0&1&1&0\end{pmatrix}$ ind

21. C[0, 1]: $\sin x$, $\cos x$ linearly dependent $\Leftrightarrow$ $\sin x \equiv a \cos x \Leftrightarrow \tan x \equiv a$, false.
Note: $\sin x$ and $\cos x$ are *functionally* dependent: $\sin^2 x + \cos^2 x = 1$.

23 and 24. n vectors in $\mathbb{R}^n$ are linearly dependent $\Leftrightarrow$ their determinant is 0.

25. Find α to make the vectors dependent: $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\4\end{pmatrix}, \begin{pmatrix}3\\\alpha\\4\end{pmatrix}$. $0 = \begin{vmatrix}1&2&3\\2&-1&\alpha\\3&4&4\end{vmatrix} = 3\cdot 11 - \alpha(-2) + 4(-5)$, $\alpha = -\frac{13}{2}$

27. $A\mathbf{c} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)(c_1, \cdots, c_n)^T = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, a linear combination of the columns. Theorem 3 is just the definition of linear independence.

29. This is the contrapositive of Problem 28 and hence logically equivalent.

31. If $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$, $\mathbf{v}_2 \perp \mathbf{v}_3$, and all 3 are nonzero, show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
▷ $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \Rightarrow 0 = \mathbf{0}\cdot\mathbf{v}_1 = c_1\mathbf{v}_1\cdot\mathbf{v}_1 + c_2\mathbf{v}_2\cdot\mathbf{v}_1 + c_3\mathbf{v}_3\cdot\mathbf{v}_1 = c_1 v_1{}^2$
$\{\mathbf{v}_2\cdot\mathbf{v}_1 = \mathbf{v}_3\cdot\mathbf{v}_1 = 0\} \Rightarrow c_1 = 0$ $\{v_1 \neq 0\}$. Similarly, $c_2 = c_3 = 0$.

In Problems 33–37, write the solution space as the span of an independent set of vectors.

33. $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3 \Rightarrow \mathbf{x} = (-x_2 - x_3, x_2, x_3) = x_2(-1, 1, 0) + x_3(-1, 0, 1)$

35. $\begin{array}{l}x_1 + 2x_2 - x_3 = 0\\2x_1 + 5x_2 + 4x_3 = 0\end{array}$ $\quad \begin{array}{l}x_1 + 2x_2 - x_3 = 0\\x_2 + 6x_3 = 0\end{array}$ $\quad \begin{array}{l}x_1 \quad -13x_3 = 0\\x_2 + 6x_3 = 0\end{array}$ $\quad \mathbf{x} = x_3(13, -6, 1)$

37. $x_1 + 2x_2 - 3x_3 + 5x_4 = 0 \Rightarrow \mathbf{x} = x_2(-2, 1, 0, 0) + x_3(3, 0, 1, 0) + x_4(-5, 0, 0, 1)$

38 and 39. (a) Show that $\mathbf{u}^\perp$ is a subspace. (b) Find 2 linearly independent vector $\mathbf{x}$ and $\mathbf{y}$ in $(1, 2, 3)^\perp$.
▷ (a) $\mathbf{v}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{u} = 0 \Rightarrow (a\mathbf{v} + b\mathbf{w})\cdot\mathbf{u} = a\mathbf{v}\cdot\mathbf{u} + b\mathbf{w}\cdot\mathbf{u} = a0 + b0 = 0$ (b) $\mathbf{x} = (-2, 1, 0)$, $\mathbf{y} = (-3, 0, 1)$
(c) $\mathbf{w} = \mathbf{x}\times\mathbf{y} = (1, 2, 3) = \mathbf{u}$ (d) $\mathbf{u}^\perp$ is the plane $\perp \mathbf{u}$, and the cross product is the vector $\perp$ to the plane.

41. Two polynomials cannot span P_2. See Problem 8.3.14.

43. Show that any subset of a set of linearly independent vectors is independent. (Pr 29 applies to finite sets.)
▷ Any combination of the vectors of a subset is a combination of the vectors of the set.

44 and 45. $mn + 1$ matrices in M_{mn} is the same as $mn + 1$ vectors in $\mathbb{R}^{mn}$ and hence dependent.

47. Show that in P_n $1, x, \ldots, x_n$ are linearly independent. ▷ $c_0 + c_1 x + \cdots + c_n x^n \equiv 0 \Leftrightarrow$ all the $c_i = 0$

49. By hypothesis, $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ with some c_i different from 0. Let k be the largest subscript for which $c_k \neq 0$ so $c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k = 0$ and $\mathbf{v}_k = (-c_1/c_k)\mathbf{v}_1 + \cdots + (-c_{k-1}/c_k)\mathbf{v}_{k-1}$.

51 and 52. Prove the results on the Wronskian and extend them to n functions.
▷ $f(x)$, $g(x)$ dependent $\Rightarrow af(x) + bg(x) \equiv 0$, a, b not both $0 \Rightarrow af'(x) + bg'(x) \equiv 0$. For each x, as a system of equations in a, b there are nontrivial solutions so the determinant W of the coefficient matrix must be zero. Conversely, suppose g is never 0. Then $(f/g)' = -W(f, g)/g^2 = 0 \Rightarrow f/g = c$. $W(f_1, \ldots, f_n) = \det(f_j{}^{(i-1)}(x))$. $\sum_{i=1}^{n} a_i f_i(x) \equiv 0 \Rightarrow \sum_{i=1}^{n} a_i f_i{}^{(j)}(x) \equiv 0$, $j = 1, \ldots, n-1$. Since there are nontrivial solutions, W must be zero. The condition for the converse is that the Wronskian of some set of $n-1$ functions, say $f_1, \ldots, f_n$, is never 0. Then
$\begin{vmatrix}f_1 & \cdots & f_{n-1} & y\\ f_1' & \cdots & f_{n-1}' & y'\\ \vdots & \vdots & & \vdots\\ f_1^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & y^{(n-1)}\end{vmatrix} = 0$ is a homogeneous linear differential equation of order $n-1$ with n solutions $(f_1, \ldots, f_{n-1}$ by equal columns, f_n by hypothesis). Hence the f_i are dependent by the extension of Th. 10.7.4.

53. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, what about $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{w}$, $\mathbf{v}+\mathbf{w}$? $\begin{vmatrix}1&1&0\\1&0&1\\0&1&1\end{vmatrix} = -2 \neq 0$ independent.

55. When are $(1, a, a^2)$, $(1, b, b^2)$, $(1, c, c^2)$ independent?
▷ $\begin{vmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{vmatrix} = (b-a)(c-a)(c-b)$ (Vandermonde) $\neq 0$ if a, b, c are distinct.

57. Extend $\begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\3\\4\end{pmatrix}$ to 3 independent vectors. $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&2\\-1&3&4\end{vmatrix} = \begin{pmatrix}-2\\-10\\7\end{pmatrix}$ is orthogonal to both $\Rightarrow$ independent

59. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ lie in the plane $ax + by + cz = d$, a, b, c not all 0. Since each vector lies on O, $d = 0$. Then $au_1 + bu_2 + cu_3 = 0$, $av_1 + bv_2 + cv_3 = 0$, $aw_1 + bw_2 + cw_3 = 0$ has nontrivial solutions for a, b, c so
$\begin{vmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{vmatrix} = \begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_3&v_3&w_3\end{vmatrix} = 0$ and so $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent.

8.5 BASIS AND DIMENSION

Basis $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for vector space V if S is linearly independent and S spans V.
In $\mathbb{R}^n$ a set of n vectors that satisfies either condition satisfies both.
S is a basis for V if each vector in V is a unique linear combination of the vectors of S.

V has Dimension n dim V $= n$: If V has a basis of n vectors, any basis of V has n vectors. dim$\{\mathbf{0}\} = 0$.

Symmetric, Skew $n \times n$ symmetric matrices have dim $\frac{1}{2}(n^2 + 1)$; skew-symmetric, dim $\frac{1}{2}(n^2 - n)$ See Pr. 26

Infinite Dimensional P has the basis $1, x, x^2, x^3, \ldots$, but a basis for C[0, 1] requires Zorn's lemma (App 6).

Problems 8.5

In Problems 1–10, determine if the set is a basis for the space H.
▷ If the number of vectors is correct then we need only show independence *or* span.

1. P_2: $1 - x^2$, x ▷ dim $P_2 = 3$; not a basis

3. P_2: $x^2 - 1$, $x^2 - 2$, $x^2 - 3$ ▷ can't span without x term

5. P_3: 3, $x^3 - 4x + 6$, x^2 ▷ dim $P_3 = 4$; not a basis

7. M_{22}: $\begin{pmatrix}a&0\\0&0\end{pmatrix}, \begin{pmatrix}0&b\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\c&0\end{pmatrix}, \begin{pmatrix}0&0\\0&d\end{pmatrix}$ ▷ basis. Independent since $abcd \neq 0$.

9. $H = \{(x, y)\colon x + y = 0\}$: $(1, -1)$ ▷ basis. 1 nonzero vector is independent

In Problems 11–14, find a basis in $\mathbb{R}^3$ for the subspace.

11. The plane $2x - y - z = 0$ ▷ $(1, 2, 0)$, $(1, 0, 2)$

13. The line $x/2 = y/3 = z/4$ ▷ $(2, 3, 4)$

15. Find the proper subspaces of $\mathbb{R}^3$. ▷ The dimension can be 1 (line on O) or 2 (plane on O).

(b) $y''' - 2y' - 4y = e^{-x}\tan x$. $\lambda^3 - 2\lambda - 4 = (\lambda-2)(\lambda^2+2\lambda+2) = (\lambda-2)[(\lambda+1)^2+1]$.
$y_1 = e^{2x}$, $y_2 = e^{-x}\cos x$, $y_3 = e^{-x}\sin x$. The equations are

$$\begin{aligned} e^{2x}c_1' &+ e^{-x}(\cos x)c_2' &+ e^{-x}(\sin x)c_3' &= 0 \\ 2e^{2x}c_1' &- e^{-x}(\cos x + \sin x)c_2' &+ e^{-x}(-\sin x + \cos x)c_3' &= 0 \\ 4e^{2x}c_1' &+ 2e^{-x}(\sin x)c_2' &- 2e^{-x}(\cos x)c_3 &= e^{-x}\tan x. \end{aligned}$$

The determinant is $W = Ce^{-\int a\,dx}$ {Problem 22} $= Ce^{\int 0\,dx} = C$. Let $x = 0$ to find $W = \begin{vmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 4 & 0 & -2 \end{vmatrix} = 10$

$$10c_1' = e^{-x}\tan x\begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}(\cos x + \sin x) & e^{-x}(-\sin x + \cos x) \end{vmatrix} = e^{-3x}\tan x\begin{vmatrix} \cos x & \sin x \\ -\cos x - \sin x & -\sin x + \cos x \end{vmatrix}$$

$= e^{-3x}\tan x$. $c_1 = \frac{1}{10}\int e^{-3x}\tan x\,dx$, nonelementary.

$$10c_2' = -e^{-x}\tan x\begin{vmatrix} e^{2x} & e^{-x}\sin x \\ 2e^{2x} & e^{-x}(-\sin x + \cos x) \end{vmatrix} = -\tan x\begin{vmatrix} 1 & \sin x \\ 2 & -\sin x + \cos x \end{vmatrix}$$

$= -\tan x(-3\sin x + \cos x) = 3(1-\cos^2 x)/\cos x - \sin x = 3\sec x - \cos x - \sin x$.
$c_2 = \frac{1}{10}[3\ln|\sec x + \tan x| - \sin x + \cos x]$.

$$10c_3' = e^{-x}\begin{vmatrix} e^{2x} & e^{-x}\cos x \\ 2e^{2x} & -e^{-x}(\cos x + \sin x) \end{vmatrix} = -\tan x\begin{vmatrix} 1 & \cos x \\ 2 & -\sin x + \cos x \end{vmatrix}$$

$= -\sec x + \cos x - 3\sin x$. $c_3 = \frac{1}{10}[-\ln|\sec x + \tan x| + \sin x + 3\cos x]$. $y_p =$

$\frac{1}{10}e^{2x}\int e^{-3x}\tan x\,dx + \frac{1}{10}e^{-x}\cos x[3\ln|\sec x + \tan x| - \sin x + \cos x] + \frac{1}{10}e^{-x}\sin x[-\ln|\sec x + \tan x| + \sin x + 3\cos x]$

$= \frac{1}{10}e^{2x}\int e^{-3x}\tan x\,dx + \frac{1}{10}e^{-x}[(3\cos x - \sin x)\ln|\sec x + \tan x| + 2\sin x\cos x + 1]$.

In Problems 31–33, solve the Euler equation.

31. Let $x = e^t$. $x^3y''' + 2x^2y'' + y = [D(D-1)(D-2) + 2D(D-1) - D + 1]y = (D^3 - D^2 - D + 1)y = 0$.
$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda+1)(\lambda-1)^2 = 0$, $\lambda = -1,1,1$. $y = ae^{-t} + (b + ct)e^t = ax^{-1} + (b + c\ln|x|)x$

33. $x^3y''' + 4x^2y'' + 3xy' + y = [D(D-1)(D-2) + 4D(D-1) + 3D + 1]y = (D^3 + D^2 + D + 1)y = 0$.
$\lambda^3 + \lambda^2 + \lambda + 1 = (\lambda+1)(\lambda^2+1)$. $y = ae^{-t} + b\cos t + c\sin t = ax^{-1} + b\cos\ln|x| + c\sin\ln|x|$

10.17 NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS: EULER'S METHODS

Euler's Method for $dy/dx = f(x,y)$, $y(x_0) = y_0$, step size h: $x_{n+1} = x_n + h$, $y_{n+1} = y_n + hf(x_n, y_n)$. Error $= O(h)$.

Improved Euler $k_1 = f(x_n, y_n)$, $k_2 = f(x + h, k_1)$. $x_{n+1} = x_n + h$, $y_{n+1} = y_n + \frac{1}{2}h(k_1 + k_2)$. Error $= O(h^2)$.

Runge-Kutta $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$, $k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$, $k_4 = f(x_n + h, y_n + hk_3)$
$x_{n+1} = x_n + h$, $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. Error $= O(h^4)$.

Problems 10.17

In Problems 1–10, use the Euler y_E, and improved Euler y_{IE} method.

▷ y_{RK} is Runge-Kutta, y is exact. We give as many decimals as needed to distinguish among the solutions.

1. $y' = x + y$, $y(0) = 1$, end $= 1$, $h = .2$. $y' - y = x$, $(y' - y)e^{-x} = xe^{-x}$, $ye^{-x} = -xe^{-x} - e^{-x} + C$. $y(0) = 0 \Rightarrow$ $1 = -1 + C$, $C = 2$. $y = 2e^x - x - 1$.

x	0	.2	.4	.6	.8	1.0
y_E	1	1.2	1.48	1.86	2.35	2.98
y_{IE}	1	1.24	1.577	2.031	2.630	3.405
y_{RK}	1	1.2428	1.58364	2.04421	2.65104	3.43650
y	1	1.24281	1.58365	2.04424	2.65108	3.43656

3. $y' = \frac{x-y}{x+y}$, $y(2) = 1$, end $= 1$, $h = -.2$. $y'(x + y) = x - y$, $yy' + (xy' + y) = x$, $\frac{1}{2}y^2 + xy = \frac{1}{2}x^2 + C$, $y(2) = 1 \Rightarrow$ $\frac{1}{2} = C$. $y = \sqrt{2x^2 + 1} - x$.

x	2	1.8	1.6	1.4	1.2	1.0
y_E	1	0.933	0.870	0.811	0.757	0.712
y_{IE}	1	0.9349594	0.8738649	0.8181110	0.7697801	0.7320708
y_{RK}	1	0.9349589	0.8738633	0.8181073	0.7697715	0.7320505
y	1	0.9349589	0.8738634	0.8181071	0.7697716	0.7320508

5. $y' = x\sqrt{1 + y^2}$, $y(1) = 0$, end $= 3$, $h = .4$. $\frac{dy}{\sqrt{1+y^2}} = x\,dx$, $\sinh^{-1} y = \frac{1}{2}x^2 + C$. $y(1) = 0 \Rightarrow 0 = \frac{1}{2} + C$.
$y = \sinh(\frac{1}{2}x^2 - \frac{1}{2})$.

x	1	1.4	1.8	2.2	2.6	3.0
y_E	0	0.4	1.00	2.02	4.01	8.3
y_{IE}	0	0.502	1.358	3.178	7.864	21.67
y_{RK}	0	0.49867	1.36925	3.3342	8.8456	27.027
y	0	0.49865	1.36929	3.3372	8.8791	27.290

7. $y' = \frac{y}{x} - 2.5x^2y^3$, $y(1) = \frac{1}{\sqrt{2}} \approx 0.7071$, end $= 2$, $h = .125$. Bernoulli: $y^{-3}y' = y^{-2}x^{-1} - 2.5x^2$, $z = y^{-2}$,
$-\frac{1}{2}z' = zx^{-1} - 2.5x^2$, $z' + 2x^{-1}z = 5x^2$, IF $= x^2$. $(z' + 2x^{-1}z)x^2 = 5x^4$, $zx^2 = x^5 + C$, $z(1) = 2 \Rightarrow 2 = 1 + C$,
$\frac{x^2}{y^2} = x^5 + 1$, $y = \frac{x}{\sqrt{x^5 + 1}}$.

x	1	1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000
y_E	0.707	0.685	0.634	0.573	0.514	0.461	0.416	0.377	0.343
y_{IE}	0.7071	0.6705	0.6196	0.5648	0.5120	0.4636	0.4205	0.3826	0.3494
y_{RK}	0.70711	0.67205	0.62097	0.56535	0.51168	0.46276	0.41938	0.38136	0.34816
y	0.70711	0.67207	0.62100	0.56537	0.51168	0.46276	0.41937	0.38135	0.34816

9. $y' = ye^x$, $y(0) = 2$, end $= 2$, $h = .2$. $dy/y = e^x dx$, $\ln|y| = e^x + C$. $y(0) = 2 \Rightarrow \ln 2 = 1 + C$. $y = 2\exp(e^x - 1)$.

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y_E	2	2.4	2.99	3.88	5.29	7.6	11.8	19	36	71	156
y_{IE}	2	2.493	3.260	4.518	6.713	10.84	19.35	38.83	89.41	241.4	781.6
y_{RK}	2	2.4956	3.2705	4.5504	6.8113	11.147	20.339	42.378	103.8	308.9	1165
y	2	2.4956	3.2706	4.5506	6.8120	11.150	20.354	42.451	104.2	311.9	1191

In Problems 11–20, use the improved Euler method to graph the solution.

▷ As some indication of accuracy, $y*$ is a Runge–Kutta with half the given step.

11. $y' = xy^2 + y^3$, $y(0) = 1$, end $= .1$ $h = 0.02$

x	0	0.02	0.04	0.06	0.08	0.10
y_{IE}	1	1.020 82	1.043 43	1.068 06	1.094 97	1.124 49
y_{RK}	1	1.020 831	1.043 455	1.068 099	1.095 032	1.124 576
$y*$	1	1.020 831	1.043 455	1.068 099	1.095 032	1.124 576

13. $y' = x + \cos(\pi y)$, $y(0) = 0$, end $= 2$, $h = .4$

x	0	0.4	0.8	1.2	1.6	2.0
y_{IE}	0	0.34	0.56	0.76	0.9771	1.342
y_{RK}	0	0.3821	0.6104	0.7811	0.9777	1.3527
$y*$	0	0.3828	0.6128	0.7821	0.9785	1.3542

15. $y' = \sin(xy)$, $y(0) = 1$, end $= 2\pi$, $h = \pi/4$

x	0	0.7854	1.5708	2.3562	3.1416	3.9270	4.7124	5.4978	6.2832
y_{IE}	1	1.28	1.65	1.46	1.09	0.87	0.79	0.93	0.95
y_{RK}	1	1.328	1.913	1.610	1.164	0.915	0.794	0.542	0.771
$y*$	1	1.330	1.920	1.618	1.162	0.872	0.704	0.594	0.515

The error of IE is 86%, of Runge is 51%, of half-Runge is only 0.5%.

17. $y' = \sqrt{y^2 - x^2}$, $y(0) = 1$, end $= 1$, $h = 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_{IE}	1	1.1048	1.2197	1.3452	1.4118	1.6304	1.7921	1.9681	2.1599	2.3694	2.5983
y_{RK}	1	1.10501	1.22019	1.34592	1.48285	1.63182	1.79391	1.97040	2.16282	2.37291	2.60265
$y*$	1	1.10501	1.22019	1.34592	1.48285	1.63182	1.79391	1.97040	2.16282	2.37291	2.60265

19. $y' = \sqrt{x + y^2}$, $y(1) = 2$, end $= 0$, $h = -0.2$.

x	1	0.8	0.6	0.4	0.2	0
y_{IE}	2	1.597	1.269	1.004	0.797	0.644
y_{RK}	2	1.594 737	1.264 495	0.998 607	0.789 908	0.635 357
$y*$	2	1.594 734	1.264 491	0.998 600	0.789 899	0.635 337

7. $f(x) = x^3 - x^2 + 2x + 3$; $a = 0$; $n = 8$

▷

n	0	1	2	3	4–8
$f^{(n)}(x)$	$x^3 - x^2 + 2x + 3$	$3x^2 - 2x + 2$	$6x - 2$	6	0
$f^{(n)}/n!$	3	2	$-2/2$	$6/6$	0

$P_8(x) = 3 + 2x - x^2 + x^3 = f(x)$

In Problems 9–13, find a bound for $|R_n(x)|$. ▷ $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

9. $f(x) = \cos x$; $a = \frac{1}{6}\pi$; $n = 5$; $x \in [0, \frac{1}{2}\pi]$ ▷ $f^{(6)}(x) = -\cos x$

In $[0, \frac{1}{6}\pi]$, $|R_5| = \left|\frac{-\cos c}{720}(x - \frac{1}{6}\pi)\right| < \frac{1}{720}(\frac{1}{6}\pi)^6 \approx 0.00003$. In $[\frac{1}{6}\pi, \frac{1}{2}\pi]$, $|R_5| < \dfrac{\frac{1}{2}\sqrt{3}}{720}(\frac{1}{3}\pi)^6 \approx 0.00159$.
Thus $|R_5| < 0.00159$ in $[0, \frac{1}{2}\pi]$.

11. $f(x) = e^x$; $a = 0$; $n = 6$; $x \in [-\ln e, \ln e]$

▷ $x \in [-1, 1]$. $f^{(7)}(x) = e^x$. $|R_6(x)| = \left|\frac{e^c}{7!}x^7\right| \le \frac{e}{5040} = 0.00053$

In Problems 13 and 14, use a Taylor polynomial of degree 4 to approximate the integral and find its error.

13. $\displaystyle\int_0^{1/2} \cos x^2 dx = \int_0^{1/2} (1 - \frac{x^4}{2}) dx = \left[x - \frac{x^5}{2 \cdot 5}\right]_0^{1/2} = .5 - \frac{.5^5}{2 \cdot 5} = 0.496875$ with error $< \dfrac{.5^9}{4! \cdot 9} = 9.04 \times 10^{-6}$

15. Find the first 5 terms of the sequence $\left\{\frac{n-2}{n}\right\}$. ▷ $\frac{1-2}{1} = -1, \frac{2-2}{2} = 0, \frac{3-2}{3} = \frac{1}{3}, \frac{4-2}{4} = \frac{1}{2}, \frac{5-2}{5} = \frac{3}{5}$

In Problems 17–18, find the general term a_n of the sequence.

	1	2	3	4	5	n
17.	$\frac{1}{8} = \frac{2 \cdot 1 - 1}{2^3}$	$\frac{3}{16} = \frac{2 \cdot 2 - 1}{2^4}$	$\frac{5}{32} = \frac{2 \cdot 3 - 1}{2^5}$	$\frac{7}{64} = \frac{2 \cdot 4 - 1}{2^6}$		$\frac{2n-1}{2^{n+2}}$

In Problems 18–24, determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

19. $\left\{\frac{-7}{n}\right\}$ ▷ Converges to $\lim_{n \to \infty} \frac{-7}{n} = 0$.

21. $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ ▷ Converges to $\lim_{n \to \infty} \frac{\ln n}{n^{1/2}} = \lim_{x \to \infty} \frac{\ln x}{x^{1/2}} \left\{\frac{\infty}{\infty}\right\} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x \to \infty} \frac{2}{x^{1/2}} = 0$.

23. $\left\{\left(1 - \frac{2}{n}\right)^n\right\}$ ▷ Converges to e^{-2} since $\lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any x.

In Problems 25–32, determine if the sequence is bounded or unbounded; increasing, decreasing, or not monotonic.

25. $\sqrt{n} \cos n$ ▷ unbounded, not monotonic

27. $2^n/(1 + 2^n)$ ▷ $= 1/(1 + 2^{-n})$. bounded, S↑ to 1

29. $(\sqrt{n} + 1)/n$ ▷ $= 1/\sqrt{n} + 1/n$. bounded, S↓ to 0

31. $(n - 7)/(n + 4)$ ▷ $= 1 - 11/(n + 4)$. bounded, S↑ to 1

In Problems 33–36, evaluate the sum.

33. $\sum_{k=2}^{10} 4^k$ ▷ $= \dfrac{4^2 - 4^{11}}{1 - 4} = 1{,}398{,}096$, a geometric progression

35. $\sum_{k=3}^{\infty}\left[\left(\frac{3}{4}\right)^k - \left(\frac{2}{5}\right)^k\right]$ ▷ $= \dfrac{(\frac{3}{4})^3}{1 - \frac{3}{4}} - \dfrac{(\frac{2}{5})^3}{1 - \frac{2}{5}} = \frac{27}{16} - \frac{8}{75} = \frac{1897}{1200}$, the difference of two geometric series

37. Write as a rational number $0.797979\ldots$ ▷ $= .79 + .0079 + .000079 + \cdots = \frac{.79}{1 - .01} = \frac{.79}{.99} = \frac{79}{99}$, geometric

In Problems 39–50, determine if the series converges or diverges.

39. $a_k = 1/(k^3 - 5)$ ▷ $k^3 a_k = 1/(1 - 5/k^3) \to 1$. CC $\sum 1/k^3$

41. $a_k = 1/(k^3 + 4)^{1/2}$ ▷ $k^{3/2} a_k = 1/(1 + 4/k^3)^{1/2} \to 1$. CC $\sum 1/k^{3/2}$

43. $a_k = 1/(k^3 + 50)^{1/3}$ ▷ $k a_k = 1/(1 + 50/k^3)^{1/3} \to 1$. DC $\sum 1/k$

45. $a_k = 10^k/k^5$ ▷ DNZ

47. $a_k = \dfrac{\sqrt{k} \ln(k+3)}{k^2 + 2}$ ▷ $k^{1.4} a_k = \dfrac{k^{1.9} \ln(k+3)}{k^2 + 2} = \dfrac{\ln(k+3)}{k^{.1} + 2k^{-1.9}} \to 0$. CC $\sum 1/k^{1.4}$

49. $a_k = e^{1/k}/k^{3/2}$ ▷ $k^{3/2} a_k = e^{1/k} \to 1$. CC $\sum 1/k^{3/2}$

In Problems 51–62, determine if the alternating series converges absolutely, conditionally, or not at all.

51. $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{50k}$ ▷ CAZ. $\sum |a_k|$ DC $\sum 1/k$. conditional

53. $\sum_{k=2}^{\infty} \dfrac{(-1)^{k+1}}{\sqrt{k(k-1)}}$ ▷ CAZ. $\sum |a_k|$ DC $\sum 1/k$. conditional

55. $\sum_{k=1}^{\infty} \dfrac{(-1)^k k^2}{k^4 + 1}$ ▷ $k^2 |a_k| = \dfrac{k^4}{k^4 + 1} = \dfrac{1}{1 + (1/k^4)} \to 1$. CC $\sum 1/k^4$. absolute

57. $\sum_{k=3}^{\infty} \dfrac{(-1)^k (k+2)(k+3)}{(k+1)^3}$ ▷ CAZ: $\dfrac{(k+2)(k+3)}{(k+1)^3} = \dfrac{(1 + 2/k)(1 + 3/k)}{k(1 + 1/k)^3} \to 0$. $|a_k|$ DC $\sum 1/k$. conditional

59. $\sum_{k=1}^{\infty} \dfrac{(-1)^k k^k}{k!}$ ▷ DNZ. In fact $\frac{k^k}{k!} = \frac{k}{1} \cdot \frac{k \cdot k \cdot \cdots \cdot k}{2 \cdot 3 \cdot \cdots \cdot k} > k$.

61. $\sum_{k=1}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)^k$ ▷ DNZ. In fact $\lim_{k \to \infty}\left(1 + \frac{1}{k}\right)^k = e$

63. With $\epsilon < 0.001$ sum $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^3}$ ▷ $\approx 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \frac{1}{7^3} - \frac{1}{8^3} + \frac{1}{9^3} = 0.9021$ with $\epsilon < \frac{1}{10^3} = 0.001$

$\approx 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{2} \cdot \frac{1}{6^3} = 0.90210$, $\epsilon < \frac{1}{2}(\frac{1}{6^3} - \frac{1}{7^3}) = 8.6 \times 10^{-4}$

65. At what time between 9 P.M. and 10 P.M. is the minute hand of a clock exactly over the hour hand?

▷ While the minute hand travels the 45 min. between 12 and 9, the hour hand advances 45/12 min. While the minute hand travels 45/12 min., the hour hand advances $45/12^2$ min., etc. The two coincide at $45/(1 - \frac{1}{12}) = \frac{540}{11}$ min. $= 39\frac{1}{11}$ min. after 9.

In Problems 67–76, find the radius and interval of convergence of the power series. ▷ $R = \lim_{k \to \infty}\left|\dfrac{a_k}{a_{k+1}}\right|$

67. $\sum_{k=0}^{\infty} \dfrac{x^k}{3^k}$ ▷ $R = \lim_{k \to \infty} \dfrac{1/3^k}{1/3^{k+1}} = \lim_{k \to \infty} 3 = 3$. $|x| = 3$: DNZ. $(-3, 3)$

69. $\sum_{k=0}^{\infty} \dfrac{x^k}{k^2 + 2}$ ▷ $R = \lim_{k \to \infty} \dfrac{(k+1)^2 + 2}{k^2 + 2} = \lim_{k \to \infty} \dfrac{(1 + 1/k)^2 + 2/k^2}{1 + 2/k^2} = 1$. $|x| = 1$: CC $\sum 1/k^2$. $[-1, 1]$

71. $\sum_{k=2}^{\infty} \dfrac{x^k}{(2 \ln k)^k}$ ▷ $R = \lim_{n \to \infty} 1/a_n{}^{1/n} = \lim_{n \to \infty} (2 \ln n) = \infty$

73. $\sum_{k=0}^{\infty} \dfrac{(3x - 5)^k}{3^k}$ ▷ $= \sum_{k=0}^{\infty} (x - \frac{5}{3})^k$. $R = 1$. $\left|x - \frac{5}{3}\right| = 1$: DNZ. $(\frac{2}{3}, \frac{8}{3})$

75. $\sum_{k=0}^{\infty} (-1)^k x^{3k}$ ▷ $R^3 = 1$, $R = 1$. $|x| = 1$: DNZ. $(-1, 1)$

In Problems 77–80, estimate the integral with specified error ϵ.

▷ The series used are CAZ with error less than the first term omitted.

77. $\epsilon < 0.00001$. $\displaystyle\int_0^{1/2} e^{-t^2} dt$ ▷ $= \displaystyle\int_0^{1/2} (1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6}) dt \approx .5 - \frac{.5^3}{3} + \frac{.5^5}{5 \cdot 2} - \frac{.5^7}{7 \cdot 6} = 0.461272$

with $\epsilon < \dfrac{.5^9}{9 \cdot 24} = 9.04 \times 10^{-6}$

79. $\epsilon < 0.001$. $\displaystyle\int_0^{1/2} t^3 e^{-t^3} dt$ ▷ $= \displaystyle\int_0^{1/2} t^3\left(1 - t^3 + \frac{t^6}{2} - \cdots\right) dt = \int_0^{1/2}\left(t^3 - t^6 + \frac{t^9}{2} - \right) dt$

$\approx \frac{1}{4}(\frac{1}{2})^4 + \frac{1}{7}(\frac{1}{2})^7 = 0.01451$ with $\epsilon < \frac{1}{20}(\frac{1}{2})^{10} = 0.00005$

81. Find the Maclaurin series for $x^2 e^x$. ▷ $= x^2 \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = \sum_{k=0}^{\infty} \dfrac{x^{k+2}}{k!}$

83. Find the Maclaurin series for $\cos^2 x$. ▷ $= \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty} (-1)^k \dfrac{(2x)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} (-1)^k \dfrac{(2x)^{2k}}{2(2k)!}$