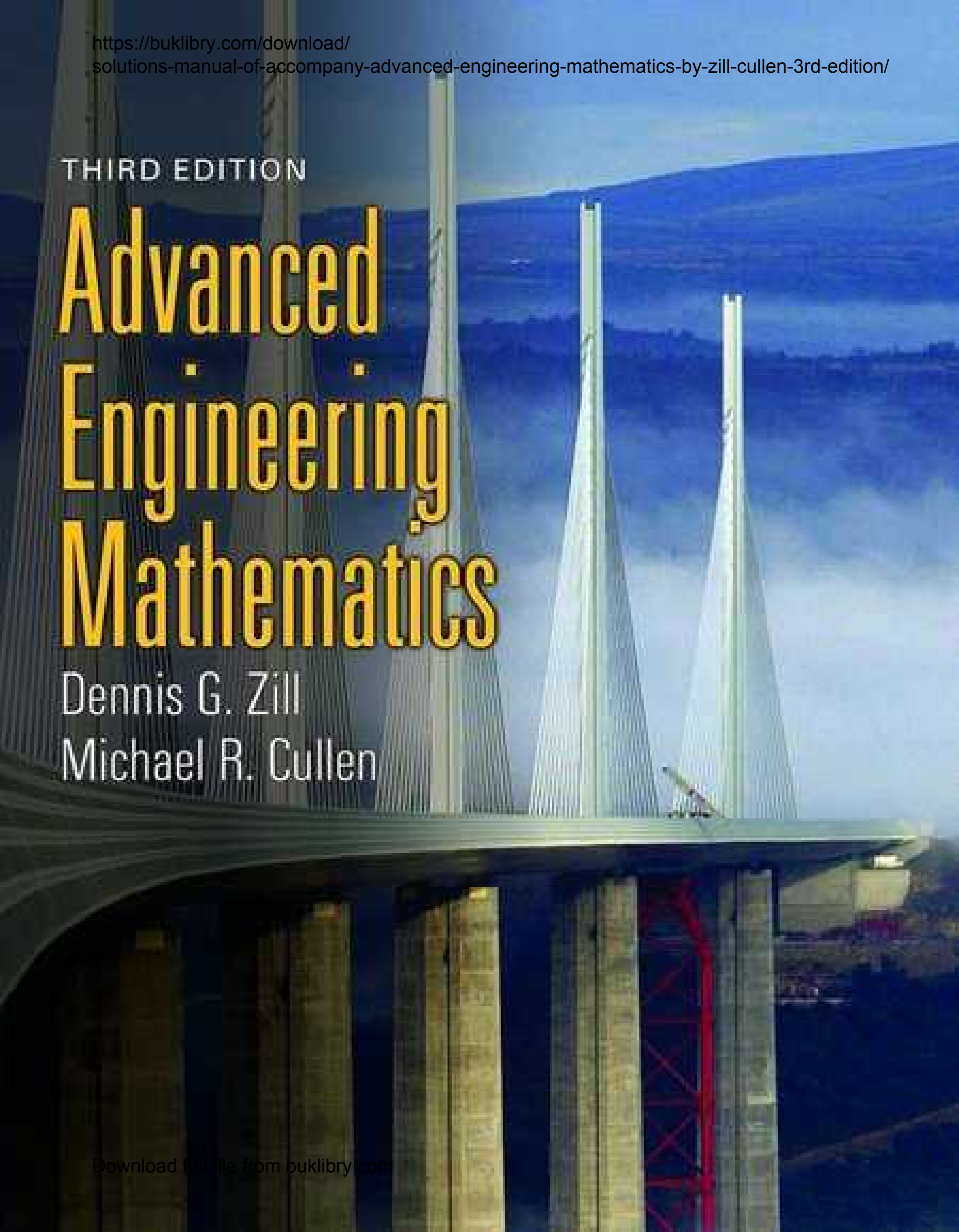
THIRD EDITION
Advanced Engineering Mathematics
Dennis G. Zill
Michael R. Cullen

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# Part V Complex Analysis



# Appendices



# Projects

Setting $h(0)=2$ we find $c_1=8\sqrt{2}/5$, so that

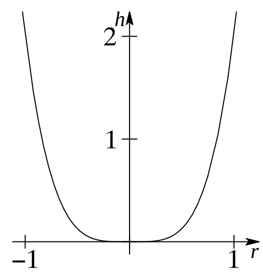
$$\frac{2}{5}\,h^{5/2}=-\frac{1}{7680}\,t+\frac{8\sqrt{2}}{5}\,,$$
$$h^{5/2}=4\sqrt{2}-\frac{1}{3072}\,t,$$

and

$$h=\left(4\sqrt{2}-\frac{1}{3072}\,t\right)^{2/5}.$$

In this case $h(4\text{ hr})=h(14{,}400\text{ s})=11.8515$ inches and $h(5\text{ hr})=h(18{,}000\text{ s})$ is not a real number. Using a CAS to solve $h(t)=0$, we see that the tank runs dry at $t\approx 17{,}378$ s $\approx 4.83$ hr. Thus, this particular conical water clock can only measure time intervals of less than 4.83 hours.

**34.** If we let $r_h$ denote the radius of the hole and $A_w=\pi[f(h)]^2$, then the differential equation $dh/dt=-k\sqrt{h}$, where $k=cA_h\sqrt{2g}/A_w$, becomes




$$\frac{dh}{dt}=-\frac{c\pi r_h^2\sqrt{2g}}{\pi[f(h)]^2}\sqrt{h}=-\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}\,.$$

For the time marks to be equally spaced, the rate of change of the height must be a constant; that is, $dh/dt=-a$. (The constant is negative because the height is decreasing.) Thus

$$-a=-\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}\,,\qquad [f(h)]^2=\frac{8cr_h^2\sqrt{h}}{a}\,,\qquad \text{and}\qquad r=f(h)=2r_h\sqrt{\frac{2c}{a}}\,h^{1/4}.$$

Solving for $h$, we have

$$h=\frac{a^2}{64c^2r_h^4}\,r^4.$$

The shape of the tank with $c=0.6$, $a=2$ ft/12 hr $=1$ ft/21,600 s, and $r_h=1/32(12)=1/384$ is shown in the above figure.

**35.** From $dx/dt=k_1x(\alpha-x)$ we obtain

$$\left(\frac{1/\alpha}{x}+\frac{1/\alpha}{\alpha-x}\right)dx=k_1\,dt$$

so that $x=\alpha c_1e^{\alpha k_1t}/(1+c_1e^{\alpha k_1t})$. From $dy/dt=k_2xy$ we obtain

$$\ln|y|=\frac{k_2}{k_1}\ln\left|1+c_1e^{\alpha k_1t}\right|+c\quad\text{or}\quad y=c_2\left(1+c_1e^{\alpha k_1t}\right)^{k_2/k_1}.$$

**36.** In tank $A$ the salt input is

$$\left(7\,\frac{\text{gal}}{\text{min}}\right)\left(2\,\frac{\text{lb}}{\text{gal}}\right)+\left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\left(14+\frac{1}{100}x_2\right)\frac{\text{lb}}{\text{min}}\,.$$

The salt output is

$$\left(3\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)+\left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{2}{25}\,x_1\,\frac{\text{lb}}{\text{min}}\,.$$

In tank $B$ the salt input is

$$\left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{1}{20}\,x_1\,\frac{\text{lb}}{\text{min}}\,.$$

The salt output is

$$\left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)+\left(4\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right)=\frac{1}{20}\,x_2\,\frac{\text{lb}}{\text{min}}\,.$$

**(e)** For each $v_0$ we want to find the smallest value of $t$ for which $r(t) = \pm 20$. Whether we look for $r(t) = -20$ or $r(t) = 20$ is determined by looking at the graphs in part (d). The total times that the bead stays on the rod is shown in the table below.

| $v_0$ | 0 | 10 | 15 | 16.1 | 17 |
|---|---|---|---|---|---|
| r | -20 | -20 | -20 | 20 | 20 |
| t | 1.55007 | 2.35494 | 3.43088 | 6.11627 | 4.22339 |

When $v_0 = 16$ the bead never leaves the rod.

**53.** Unlike the derivation given in Section 3.8 in the text, the weight $mg$ of the mass $m$ does not appear in the net force since the spring is not stretched by the weight of the mass when it is in the equilibrium position (i.e. there is no $mg - ks$ term in the net force). The only force acting on the mass when it is in motion is the restoring force of the spring. By Newton's second law,

$$m\,\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

**54.** The force of kinetic friction opposing the motion of the mass in $\mu N$, where $\mu$ is the coefficient of sliding friction and $N$ is the normal component of the weight. Since friction is a force opposite to the direction of motion and since $N$ is pointed directly downward (it is simply the weight of the mass), Newton's second law gives, for motion to the right $(x' > 0)$ ,

$$m\,\frac{d^2x}{dt^2} = -kx - \mu mg,$$

and for motion to the left $(x' < 0)$,

$$m\,\frac{d^2x}{dt^2} = -kx + \mu mg.$$

Traditionally, these two equations are written as one expression

$$m\,\frac{d^2x}{dt^2} + f_x\,\mathrm{sgn}(x') + kx = 0,$$

where $f_k = \mu mg$ and

$$\mathrm{sgn}(x') = \begin{cases} 1, & x' > 0 \\ -1, & x' < 0. \end{cases}$$

Letting $t = \frac{2}{3}\alpha x^{3/2}$ or $\alpha x^{3/2} = \frac{3}{2}t$ this differential equation becomes

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2 w''(t) + tw'(t) + \left(t^2 - \frac{1}{9}\right)w(t)\right] = 0, \qquad t > 0.$$

**35.** **(a)** By Problem 34, a solution of Airy's equation is $y = x^{1/2}w(\frac{2}{3}\alpha x^{3/2})$, where

$$w(t) = c_1 J_{1/3}(t) + c_2 J_{-1/3}(t)$$

is a solution of Bessel's equation of order $\frac{1}{3}$. Thus, the general solution of Airy's equation for $x > 0$ is

$$y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) = c_1 x^{1/2} J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2 x^{1/2} J_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right).$$

**(b)** Airy's equation, $y'' + \alpha^2 xy = 0$, has the form of (18) in the text with

$$\begin{aligned} 1 - 2a = 0 &\implies a = \frac{1}{2} \\ 2c - 2 = 1 &\implies c = \frac{3}{2} \\ b^2c^2 = \alpha^2 &\implies b = \frac{2}{3}\alpha \\ a^2 - p^2c^2 = 0 &\implies p = \frac{1}{3}. \end{aligned}$$

Then, by (19) in the text,

$$y = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right)\right].$$

**36.** The general solution of the differential equation is

$$y(x) = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x).$$

In order to satisfy the conditions that $\lim_{x\to 0^+} y(x)$ and $\lim_{x\to 0^+} y'(x)$ are finite we are forced to define $c_2 = 0$. Thus, $y(x) = c_1 J_0(\alpha x)$. The second boundary condition, $y(2) = 0$, implies $c_1 = 0$ or $J_0(2\alpha) = 0$. In order to have a nontrivial solution we require that $J_0(2\alpha) = 0$. From Table 5.1, the first three positive zeros of $J_0$ are found to be

$$2\alpha_1 = 2.4048, \quad 2\alpha_2 = 5.5201, \quad 2\alpha_3 = 8.6537$$

and so $\alpha_1 = 1.2024$, $\alpha_2 = 2.7601$, $\alpha_3 = 4.3269$. The eigenfunctions corresponding to the eigenvalues $\lambda_1 = \alpha_1^2$, $\lambda_2 = \alpha_2^2$, $\lambda_3 = \alpha_3^2$ are $J_0(1.2024x)$, $J_0(2.7601x)$, and $J_0(4.3269x)$.

**37.** **(a)** The differential equation $y'' + (\lambda/x)y = 0$ has the form of (18) in the text with

$$\begin{aligned} 1 - 2a = 0 &\implies a = \frac{1}{2} \\ 2c - 2 = -1 &\implies c = \frac{1}{2} \\ b^2c^2 = \lambda &\implies b = 2\sqrt{\lambda} \\ a^2 - p^2c^2 = 0 &\implies p = 1. \end{aligned}$$

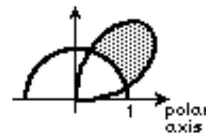
Then, by (19) in the text,

$$y = x^{1/2}[c_1 J_1(2\sqrt{\lambda x}\,) + c_2 Y_1(2\sqrt{\lambda x}\,)].$$

**(b)** We first note that $y = J_1(t)$ is a solution of Bessel's equation, $t^2y'' + ty' + (t^2 - 1)y = 0$, with $\nu = 1$. That is,

$$t^2 J_1''(t) + tJ_1'(t) + (t^2 - 1)J_1(t) = 0,$$

**20.** Solving $1 = 2\sin 2\theta$, we obtain $\sin 2\theta = 1/2$ or $\theta = \pi/12$ and $\theta = 5\pi/12$.

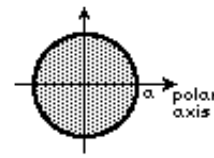



$$I_y = \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} x^2\sec^2\theta\, r\,dr\,d\theta = \int_{\pi/12}^{5\pi/12}\int_1^{2\sin 2\theta} r^3\,dr\,d\theta$$

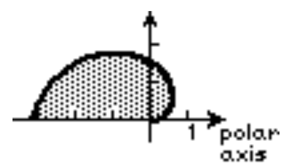
$$= \int_{\pi/12}^{5\pi/12} \frac{1}{4}r^4\Big|_1^{2\sin 2\theta} d\theta = 4\int_{\pi/12}^{5\pi/12}\sin^4 2\theta\,d\theta = 2\left(\frac{3}{4}\theta - \frac{1}{4}\sin 4\theta + \frac{1}{32}\sin 8\theta\right)\Big|_{\pi/12}^{5\pi/12}$$

$$= 2\left[\left(\frac{5\pi}{16} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{64}\right) - \left(\frac{\pi}{16} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{64}\right)\right] = \frac{8\pi + 7\sqrt{3}}{16}$$

**21.** From the solution to Problem 17, $I_x = k\pi a^4/4$. By symmetry, $I_y = I_x$. Thus $I_0 = k\pi a^4/2$.



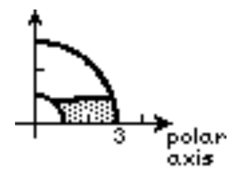

**22.** The density is $\rho = kr$.




$$I_0 = \int_0^\pi\int_0^\theta r^2(kr)r\,dr\,d\theta = k\int_0^\pi\int_0^\theta r^4\,dr\,d\theta = k\int_0^\pi \frac{1}{5}r^5\Big|_0^\theta d\theta$$

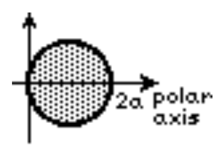
$$= \frac{1}{5}k\int_0^\pi \theta^5\,d\theta = \frac{1}{5}k\left(\frac{1}{6}\theta^6\right)\Big|_0^\pi = \frac{k\pi^6}{30}$$

**23.** The density is $\rho = k/r$.



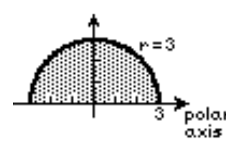

$$I_0 = \int_1^3\int_0^{1/r} r^2\,\frac{k}{r}\,r\,d\theta\,dr = k\int_1^3\int_0^{1/r} r^2\,d\theta\,dr = k\int_1^3 r^2\left(\frac{1}{r}\right)dr = k\left(\frac{1}{2}r^2\right)\Big|_1^3 = 4k$$

**24.** $$I_0 = \int_0^\pi\int_0^{2a\cos\theta} r^2kr\,dr\,d\theta = k\int_0^\pi \frac{1}{4}r^4\Big|_0^{2a\cos\theta} d\theta = 4ka^4\int_0^\pi \cos^4\theta\,d\theta$$

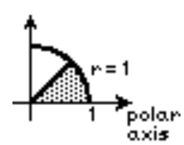



$$= 4ka^4\left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Big|_0^\pi = 4ka^4\left(\frac{3\pi}{8}\right) = \frac{3k\pi a^4}{2}$$

**25.** $$\int_{-3}^3\int_0^{\sqrt{9-x^2}}\sqrt{x^2+y^2}\,dy\,dx = \int_0^\pi\int_0^3 |r|r\,dr\,d\theta = \int_0^\pi \frac{1}{3}r^3\Big|_0^3 d\theta = 9\int_0^\pi d\theta = 9\pi$$



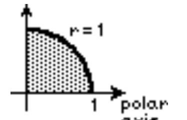

**26.** $$\int_0^{\sqrt{2}/2}\int_y^{\sqrt{1-y^2}} \frac{y^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/4}\int_0^1 \frac{r^2\sin^2\theta}{|r|}\,r\,dr\,d\theta$$




$$= \int_0^{\pi/4}\int_0^1 r^2\sin^2\theta\,dr\,d\theta = \int_0^{\pi/4}\frac{1}{3}r^3\sin^2\theta\Big|_0^1 d\theta = \frac{1}{3}\int_0^{\pi/4}\sin^2\theta\,d\theta$$

$$= \frac{1}{3}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{\pi - 2}{24}$$

**27.** $$\int_0^1\int_0^{\sqrt{1-y^2}} e^{x^2+y^2}dx\,dy = \int_0^{\pi/2}\int_0^1 e^{r^2}r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}e^{r^2}\Big|_0^1 d\theta$$




$$= \frac{1}{2}\int_0^{\pi/2}(e-1)\,d\theta = \frac{\pi(e-1)}{4}$$

$$= c_1 \begin{pmatrix} \cos t + \sin t \\ -2\sin t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix} e^{-t}.$$

**19.** The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$. This leads to the system

$$e^t = b_0 + b_1$$
$$e^{6t} = b_0 + 6b_1,$$

which has the solution $b_0 = \frac{6}{5}e^t - \frac{1}{5}e^{6t}$ and $b_1 = -\frac{1}{5}e^t + \frac{1}{5}e^{6t}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} \frac{4}{5}e^t + \frac{1}{5}e^{6t} & \frac{2}{5}e^t - \frac{2}{5}e^{6t} \\ \frac{2}{5}e^t - \frac{2}{5}e^{6t} & \frac{1}{5}e^t + \frac{4}{5}e^{6t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} \frac{4}{5}e^t + \frac{1}{5}e^{6t} & \frac{2}{5}e^t - \frac{2}{5}e^{6t} \\ \frac{2}{5}e^t - \frac{2}{5}e^{6t} & \frac{1}{5}e^t + \frac{4}{5}e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix} e^t + c_1 \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix} e^t + c_2 \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \end{pmatrix} e^{6t} \\ &= \left(\frac{2}{5}c_1 + \frac{1}{5}c_2\right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + \left(\frac{1}{5}c_1 - \frac{2}{5}c_2\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t}. \end{aligned}$$

**20.** The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. This leads to the system

$$e^{2t} = b_0 + 2b_1$$
$$e^{3t} = b_0 + 3b_1,$$

which has the solution $b_0 = 3e^{2t} - 2e^{3t}$ and $b_1 = -e^{2t} + e^{3t}$. Then

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{3t} \\ &= (c_1 - c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + (-c_1 + 2c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}. \end{aligned}$$

# EXERCISES 15.2

## Applications of the Laplace Transform

**1.** The boundary-value problem is

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,, \quad 0 < x < L, \quad t > 0,$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$
$$u(x,0) = A\sin\frac{\pi}{L}x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - \left(\frac{s}{a}\right)^2 U = -\frac{s}{a^2}A\sin\frac{\pi}{L}x.$$

Using undetermined coefficients we obtain

$$U(x,s) = c_1\cosh\frac{s}{a}x + c_2\sinh\frac{s}{a}x + \frac{As}{s^2 + a^2\pi^2/L^2}\sin\frac{\pi}{L}x.$$

The transformed boundary conditions, $U(0,s) = 0$, $U(L,s) = 0$ give in turn $c_1 = 0$ and $c_2 = 0$. Therefore

$$U(x,s) = \frac{As}{s^2 + a^2\pi^2/L^2}\sin\frac{\pi}{L}x$$

and

$$u(x,t) = A\mathscr{L}^{-1}\left\{\frac{s}{s^2 + a^2\pi^2/L^2}\right\}\sin\frac{\pi}{L}x = A\cos\frac{a\pi}{L}t\sin\frac{\pi}{L}x.$$

**2.** The transformed equation is

$$\frac{d^2U}{dx^2} - s^2U = -2\sin\pi x - 4\sin 3\pi x$$

and so

$$U(x,s) = c_1\cosh sx + c_2\sinh sx + \frac{2}{s^2+\pi^2}\sin\pi x + \frac{4}{s^2+9\pi^2}\sin 3\pi x.$$

The transformed boundary conditions, $U(0,s) = 0$ and $U(1,s) = 0$ give $c_1 = 0$ and $c_2 = 0$. Thus

$$U(x,s) = \frac{2}{s^2+\pi^2}\sin\pi x + \frac{4}{s^2+9\pi^2}\sin 3\pi x$$

and

$$\begin{aligned} u(x,t) &= 2\mathscr{L}^{-1}\left\{\frac{1}{s^2+\pi^2}\right\}\sin\pi x + 4\mathscr{L}^{-1}\left\{\frac{1}{s^2+9\pi^2}\right\}\sin 3\pi x \\ &= \frac{2}{\pi}\sin\pi t\sin\pi x + \frac{4}{3\pi}\sin 3\pi t\sin 3\pi x. \end{aligned}$$

**3.** The solution of

$$a^2\frac{d^2U}{dx^2} - s^2U = 0$$

is in this case

$$U(x,s) = c_1e^{-(x/a)s} + c_2e^{(x/a)s}.$$

# 19 Series and Residues

## EXERCISES 19.1

### Sequences and Series

**1.** $5i,\ -5,\ -5i,\ 5,\ 5i$

**2.** $2-i,\ 1,\ 2+i,\ 3,\ 2-i$

**3.** $0,\ 2,\ 0,\ 2,\ 0$

**4.** $1+i,\ 2i,\ -2+2i,\ -4,\ -4-4i$

**5.** Converges. To see this write the general term as $\dfrac{3i+2/n}{1+i}$.

**6.** Converges. To see this write the general term as $\left(\dfrac{2}{5}\right)^n \dfrac{1+n2^{-n}i}{1+3n5^{-n}i}$.

**7.** Converges. To see this write the general term as $\dfrac{(i+2/n)^2}{i}$.

**8.** Diverges. To see this consider the term $\dfrac{n}{n+1}\,i^n$ and take $n$ to be an odd positive integer.

**9.** Diverges. To see this write the general term as $\sqrt{n}\left(1+\dfrac{1}{\sqrt{n}}\,i^n\right)$.

**10.** Converges. The real part of the general term converges to 0 and the imaginary part of the general term converges to $\pi$.

**11.** $\text{Re}(z_n)=\dfrac{8n^2+n}{4n^2+1}\to 2$ as $n\to\infty$, and $\text{Im}(z_n)=\dfrac{6n^2-4n}{4n^2+1}\to\dfrac{3}{2}$ as $n\to\infty$.

**12.** Write $z_n=\left(\dfrac{1}{4}+\dfrac{1}{4}i\right)^n$ in polar form as $z_n=\left(\dfrac{\sqrt{2}}{4}\right)^n\cos n\theta+i\left(\dfrac{\sqrt{2}}{4}\right)^n\sin n\theta$. Now

$$\text{Re}(z_n)=\left(\frac{\sqrt{2}}{4}\right)^n\cos n\theta\to 0 \text{ as } n\to\infty \quad\text{and}\quad \text{Im}(z_n)=\left(\frac{\sqrt{2}}{4}\right)^n\sin n\theta\to 0 \text{ as } n\to\infty$$

since $\sqrt{2}/4<1$.

**13.** $S_n=\dfrac{1}{1+2i}-\dfrac{1}{2+2i}+\dfrac{1}{2+2i}-\dfrac{1}{3+2i}+\dfrac{1}{3+2i}-\dfrac{1}{4+2i}+\cdots+\dfrac{1}{n+2i}-\dfrac{1}{n+1+2i}=\dfrac{1}{1+2i}-\dfrac{1}{n+1+2i}$

Thus, $\lim\limits_{n\to\infty}S_n=\dfrac{1}{1+2i}=\dfrac{1}{5}-\dfrac{2}{5}i$.

**14.** By partial fractions, $\dfrac{i}{k(k+1)}=\dfrac{i}{k}-\dfrac{i}{k+1}$ and so

$$S_n=i-\frac{i}{2}+\frac{i}{2}-\frac{i}{3}+\frac{i}{3}-\frac{i}{4}+\cdots+\frac{i}{n}-\frac{i}{n+1}=i-\frac{i}{n+1}.$$

Thus $\lim\limits_{n\to\infty}S_n=i$.