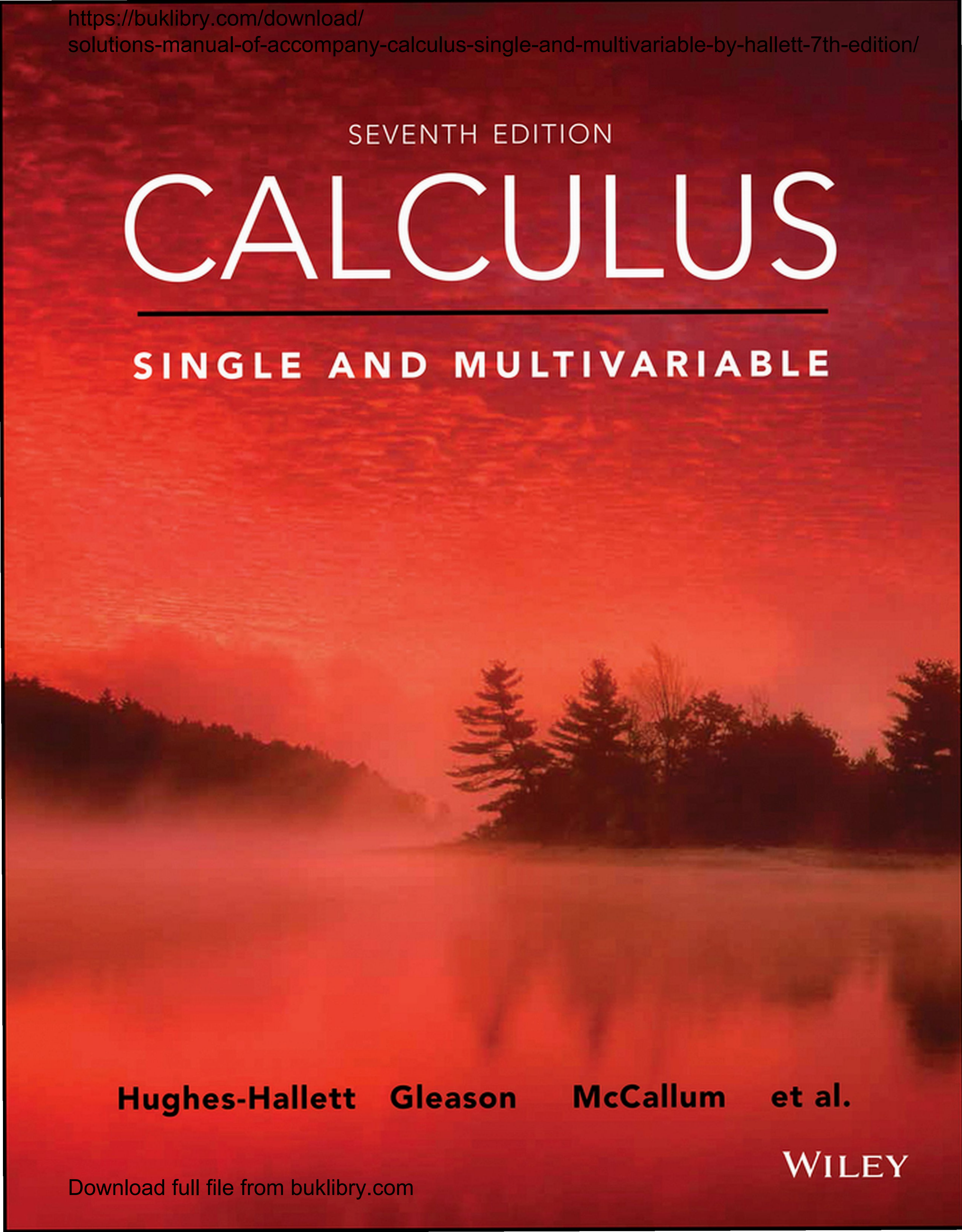

SEVENTH EDITION

CALCULUS

SINGLE AND MULTIVARIABLE

Hughes-Hallett Gleason McCallum et al.

WILEY

CONTENTS

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10 APPROXIMATING FUNCTIONS USING SERIES 513

11 DIFFERENTIAL EQUATIONS 561

12 FUNCTIONS OF SEVERAL VARIABLES 651

13 A FUNDAMENTAL TOOL: VECTORS 701

14 DIFFERENTIATING FUNCTIONS OF SEVERAL VARIABLES 739

15 OPTIMIZATION: LOCAL AND GLOBAL EXTREMA 805

16 INTEGRATING FUNCTIONS OF SEVERAL VARIABLES 839

Chapter One

FOUNDATION FOR CALCULUS: FUNCTIONS AND LIMITS

Contents

1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In 2015, Boston, Massachusetts, had the highest annual snowfall, 110.6 inches, since recording started in 1872. Table 1.1 shows one 14-day period in which the city broke another record with a total of 64.4 inches.[1]

Table 1.1 *Daily snowfall in inches for Boston, January 27 to February 9, 2015*

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Snowfall	22.1	0.2	0	0.7	1.3	0	16.2	0	0	0.8	0	0.9	7.4	14.8

You may not have thought of something so unpredictable as daily snowfall as being a function, but it *is* a function of day, because each day gives rise to one snowfall total. There is no formula for the daily snowfall (otherwise we would not need a weather bureau), but nevertheless the daily snowfall in Boston does satisfy the definition of a function: Each day, t, has a unique snowfall, S, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the snowfall example, the domain is the set of days $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ and the range is the set of daily snowfalls $\{0, 0.2, 0.7, 0.8, 0.9, 1.3, 7.4, 14.8, 16.2, 22.1\}$. We call the function f and write $S = f(t)$. Notice that a function may have identical outputs for different inputs (Days 8 and 9, for example).

Some quantities, such as a day or date, are *discrete*, meaning they take only certain isolated values (days must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily snowfall in Boston can be represented by the graph in Figure 1.1, as well as by Table 1.1.

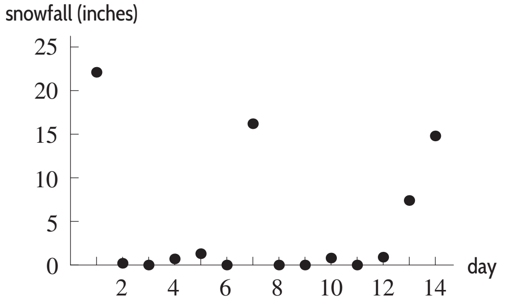

Figure 1.1: Boston snowfall, starting January 27, 2015

As another example of a function, consider the snowy tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine

[1] http://w2.weather.gov/climate/xmacis.php?wfo=box. Accessed June 2015.

Inverse Functions

On August 26, 2005, the runner Kenenisa Bekele[30] of Ethiopia set a world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.14, where $t = f(d)$ is the number of seconds Bekele took to complete the first d meters of the race. For example, Bekele ran the first 4000 meters in 629.98 seconds, so $f(4000) = 629.98$. The function f was useful to athletes planning to compete with Bekele.

Let us now change our point of view and ask for distances rather than times. If we ask how far Bekele ran during the first 629.98 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function*[31] of f. We write $f^{-1}(629.98) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Bekele ran during the first t seconds of his race. See Table 1.15, which contains values of f^{-1}.

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \le d \le 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \le t \le 1577.53$, which is the domain of f^{-1}.

Table 1.14 *Bekele's running time*

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

Table 1.15 *Distance run by Bekele*

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
315.63	2000
629.98	4000
944.66	6000
1264.63	8000
1577.53	10000

Which Functions Have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why it does not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.43.)

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Bekele time function did have an inverse is that each running time, t, corresponds to a unique distance, d.

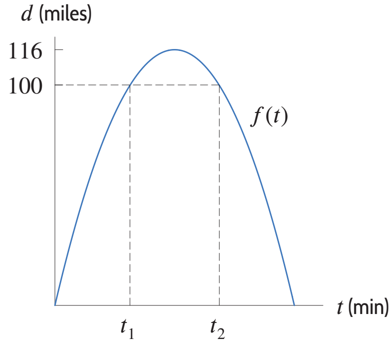

Figure 1.43: Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

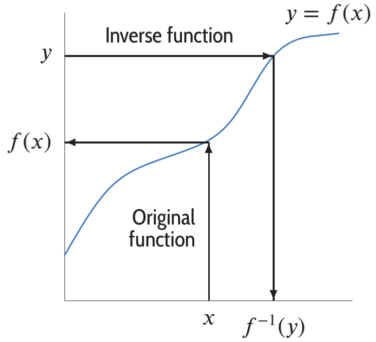

Figure 1.44: A function which has an inverse

[30] kenenisabekelle.com/, accessed January 11, 2011.
[31] The notation f^{-1} represents the inverse function, which is not the same as the reciprocal, $1/f$.

48. **(a)** If $f(x) = ax^2 + bx + c$, what can you say about the values of a, b, and c if:
   (i) $(1, 1)$ is on the graph of $f(x)$?
   (ii) $(1, 1)$ is the vertex of the graph of $f(x)$? (Hint: The axis of symmetry is $x = -b/(2a)$.)
   (iii) The y-intercept of the graph is $(0, 6)$?

(b) Find a quadratic function satisfying all three conditions.

49. A box of fixed volume V has a square base with side length x. Write a formula for the height, h, of the box in terms of x and V. Sketch a graph of h versus x.

50. A closed cylindrical can of fixed volume V has radius r.

(a) Find the surface area, S, as a function of r.
(b) What happens to the value of S as $r \to \infty$?
(c) Sketch a graph of S against r, if $V = 10$ cm^3.

51. The DuBois formula relates a person's surface area s, in m^2, to weight w, in kg, and height h, in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

(a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?
(b) What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m^2?
(c) For people of fixed weight 70 kg, solve for h as a function of s. Simplify your answer.

52. According to *Car and Driver*, an Alfa Romeo going at 70 mph requires 150 feet to stop.[56] Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph.

53. Poiseuille's Law gives the rate of flow, R, of a gas through a cylindrical pipe in terms of the radius of the pipe, r, for a fixed drop in pressure between the two ends of the pipe.

(a) Find a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.
(b) If $R = 400$ cm^3/sec in a pipe of radius 3 cm for a certain gas, find a formula for the rate of flow of that gas through a pipe of radius r cm.
(c) What is the rate of flow of the same gas through a pipe with a 5 cm radius?

54. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time t seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?

55. The height of an object above the ground at time t is given by

$$s = v_0 t - \frac{g}{2}t^2,$$

where v_0 is the initial velocity and g is the acceleration due to gravity.

(a) At what height is the object initially?
(b) How long is the object in the air before it hits the ground?
(c) When will the object reach its maximum height?
(d) What is that maximum height?

56. The rate, R, at which a population in a confined space increases is proportional to the product of the current population, P, and the difference between the carrying capacity, L, and the current population. (The carrying capacity is the maximum population the environment can sustain.)

(a) Write R as a function of P.
(b) Sketch R as a function of P.

■ In Problems 57–61, the length of a plant, L, is a function of its mass, M. A unit increase in a plant's mass stretches the plant's length more when the plant is small, and less when the plant is large.[57] Assuming $M > 0$, decide if the function agrees with the description.

57. $L = 2M^{1/4}$

58. $L = 0.2M^3 + M^4$

59. $L = 2M^{-1/4}$

60. $L = \dfrac{4(M+1)^2 - 1}{(M+1)^2}$

61. $L = \dfrac{10(M+1)^2 - 1}{(M+1)^3}$

■ In Problems 62–64, find all horizontal and vertical asymptotes for each rational function.

62. $f(x) = \dfrac{5x - 2}{2x + 3}$

63. $f(x) = \dfrac{x^2 + 5x + 4}{x^2 - 4}$

64. $f(x) = \dfrac{5x^3 + 7x - 1}{x^3 - 27}$

65. For each function, fill in the blanks in the statements:
$f(x) \to$ __________ as $x \to -\infty$,
$f(x) \to$ __________ as $x \to +\infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$
(b) $f(x) = \dfrac{3x^2 - 5x + 2}{2x^2 - 8}$
(c) $f(x) = e^x$

66. A rational function $y = f(x)$ is graphed in Figure 1.95. If $f(x) = g(x)/h(x)$ with $g(x)$ and $h(x)$ both quadratic functions, give possible formulas for $g(x)$ and $h(x)$.

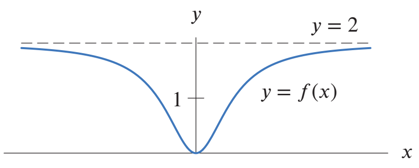

Figure 1.95

[56] http://www.caranddriver.com/alfa-romeo/4c. Accessed February 2016.
[57] Niklas, K. and Enquist, B., "Invariant scaling relationships for interspecific plant biomass production rates and body size", *PNAS*, Feb 27, 2001.

Solution If we try to evaluate at $x = 3$, we get $4/0$ which is undefined. Figure 1.129 shows that as x approaches 3 from the right, the function becomes arbitrarily large, and as x approaches 3 from the left, the function becomes arbitrarily large but negative, so this limit does not exist.

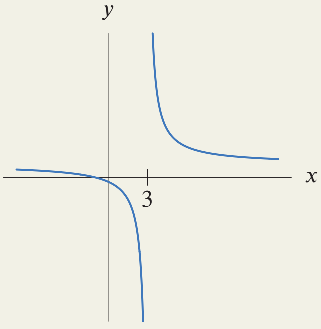

Figure 1.129: Limit of $y = (x + 1)/(x - 3)$ does not exist at $x = 3$

The limit in Example 1 does not exist because as x approaches 3, the denominator gets close to zero and the numerator gets close to 4. This means we are dividing a number close to 4 by a smaller and smaller number, resulting in a larger and larger number. This observation holds in general: for continuous functions, if $g(c) = 0$ but $f(c) \neq 0$, then $\lim_{x\to c} f(x)/g(x)$ does not exist.

Limits of the Form $0/0$ and Holes in Graphs

In Example 4 of Section 1.7 we saw that when both $f(c) = 0$ and $g(c) = 0$, so we have a limit of the form $0/0$, the limit can exist. We now explore limits of this form in more detail.

Example 2 Evaluate the following limit or explain why it does not exist:

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3}.$$

Solution If we try to evaluate at $x = 3$, we get $0/0$ which is undefined. Figure 1.130 suggests that as x approaches 3, the function gets close to 5, which suggests the limit is 5.

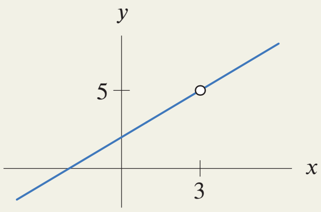

Figure 1.130: Graph of $y = (x^2 - x - 6)/(x - 3)$ is the same as the graph of $y = x + 2$ except at $x = 3$

This limit is similar to the one we saw in Example 4 of Section 1.7, so we check it algebraically using a similar method. Since the numerator factors as $x^2 - x - 6 = (x - 3)(x + 2)$ and $x \neq 3$ in the limit, we can cancel the common factor $x - 3$. We have:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3} &= \lim_{x\to 3} \frac{(x - 3)(x + 2)}{x - 3} && \text{Factoring the numerator} \\
&= \lim_{x\to 3}(x + 2) && \text{Canceling } (x - 3) \text{ since } x \neq 3 \\
&= 3 + 2 = 5 && \text{Substituting } x = 3 \text{ since } x + 2 \text{ is continuous}
\end{aligned}$$

This identity shows us how the hyperbolic functions got their name. Suppose (x, y) is a point in the plane and $x = \cosh t$ and $y = \sinh t$ for some t. Then the point (x, y) lies on the hyperbola $x^2 - y^2 = 1$.

Extending the analogy to the trigonometric functions, we define

Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Derivatives of Hyperbolic Functions

We calculate the derivatives using the fact that $\frac{d}{dx}(e^x) = e^x$. The results are again reminiscent of the trigonometric functions. For example,

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

We find $\frac{d}{dx}(\sinh x)$ similarly, giving the following results:

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sinh x) = \cosh x$$

Example 3 Compute the derivative of $\tanh x$.

Solution Using the quotient rule gives

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = \frac{1}{\cosh^2 x}.$$

Exercises and Problems for Section 3.8

Online Resource: Additional Problems for Section 3.8

EXERCISES

In Exercises 1–11, find the derivative of the function.

1. $y = \sinh(3z + 5)$
2. $y = \cosh(2x)$
3. $g(t) = \cosh^2 t$
4. $f(t) = \cosh(\sinh t)$
5. $f(t) = t^3 \sinh t$
6. $y = \cosh(3t)\sinh(4t)$
7. $y = \tanh(12 + 18x)$
8. $f(t) = \cosh(e^{t^2})$
9. $g(\theta) = \ln(\cosh(1 + \theta))$
10. $f(y) = \sinh(\sinh(3y))$
11. $f(t) = \cosh^2 t - \sinh^2 t$
12. Show that $d(\sinh x)/dx = \cosh x$.
13. Show that $\sinh 0 = 0$.
14. Show that $\sinh(-x) = -\sinh(x)$.

In Exercises 15–16, simplify the expressions.

15. $\cosh(\ln t)$
16. $\sinh(\ln t)$

Solution

(a) Each term is twice the previous term plus one; for example, $7 = 2 \cdot 3 + 1$ and $63 = 2 \cdot 31 + 1$. Thus, a recursive definition is

$$s_n = 2s_{n-1} + 1 \text{ for } n > 1 \text{ and } s_1 = 1.$$

There are other ways to define the sequence recursively. We might notice, for example, that the differences of consecutive terms are powers of 2. Thus, we could also use

$$s_n = s_{n-1} + 2^{n-1} \quad \text{for } n > 1 \text{ and } s_1 = 1.$$

(b) We recognize the terms as the squares of the positive integers, but we are looking for a recursive definition which relates consecutive terms. We see that

$$\begin{aligned} s_2 &= s_1 + 3 \\ s_3 &= s_2 + 5 \\ s_4 &= s_3 + 7 \\ s_5 &= s_4 + 9, \end{aligned}$$

so the differences between consecutive terms are consecutive odd integers. The difference between s_n and s_{n-1} is $2n - 1$, so a recursive definition is

$$s_n = s_{n-1} + 2n - 1, \text{ for } n > 1 \text{ and } s_1 = 1.$$

Recursively defined sequences, sometimes called recurrence relations, are powerful tools used frequently in computer science, as well as in differential equations. Finding a formula for the general term can be surprisingly difficult.

Convergence of Sequences

The limit of a sequence s_n as $n \to \infty$ is defined the same way as the limit of a function $f(x)$ as $x \to \infty$; see also Problem 80 (available online).

> The sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ has a **limit** L, written $\lim_{n\to\infty} s_n = L$, if s_n is as close to L as we please whenever n is sufficiently large. If a limit, L, exists, we say the sequence **converges** to its limit L. If no limit exists, we say the sequence **diverges**.

To calculate the limit of a sequence, we use what we know about the limits of functions, including the properties in Theorem 1.2 and the following facts:

- The sequence $s_n = x^n$ converges to 0 if $|x| < 1$ and diverges if $|x| > 1$
- The sequence $s_n = 1/n^p$ converges to 0 if $p > 0$

Example 5 Do the following sequences converge or diverge? If a sequence converges, find its limit.

(a) $s_n = (0.8)^n$ (b) $s_n = \dfrac{1 - e^{-n}}{1 + e^{-n}}$ (c) $s_n = \dfrac{n^2 + 1}{n}$ (d) $s_n = 1 + (-1)^n$

Solution

(a) Since $0.8 < 1$, the sequence converges by the first fact and the limit is 0.

(b) Since $e^{-1} < 1$, we have $\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} (e^{-1})^n = 0$ by the first fact. Thus, using the properties of limits from Section 1.8, we have

$$\lim_{n\to\infty} \frac{1 - e^{-n}}{1 + e^{-n}} = \frac{1 - 0}{1 + 0} = 1.$$

(c) Since $(n^2 + 1)/n$ grows without bound as $n \to \infty$, the sequence s_n diverges.

(d) Since $(-1)^n$ alternates in sign, the sequence alternates between 0 and 2. Thus the sequence s_n diverges, since it does not get close to any fixed value.

16. Match the surfaces (a)–(e) in Figure 12.52 with the contour diagrams (I)–(V) in Figure 12.53.

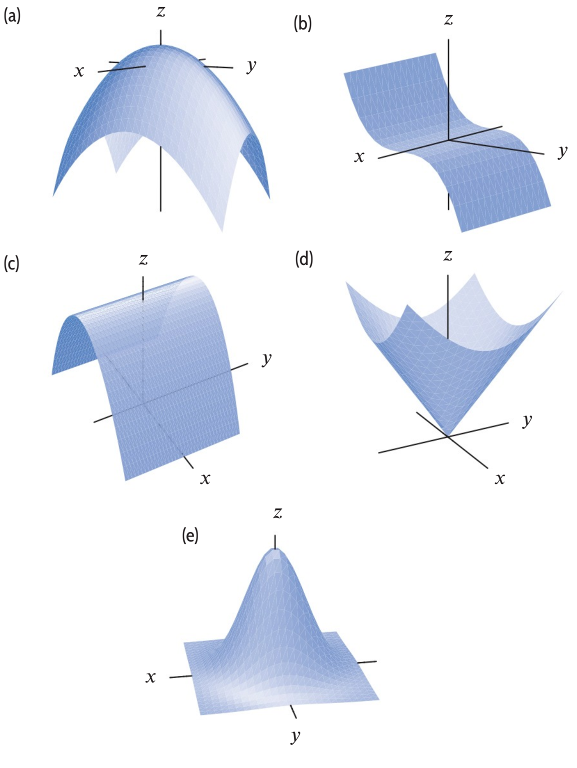

Figure 12.52

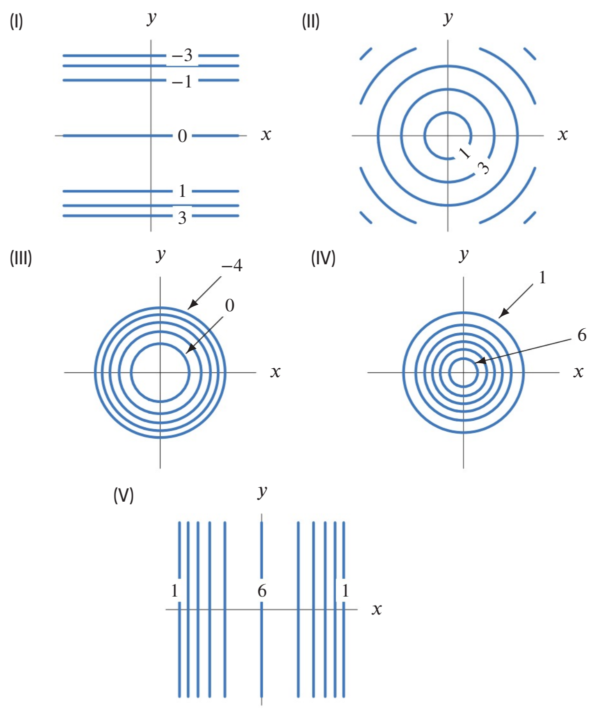

Figure 12.53

17. Figure 12.54 shows the contour diagram of $z = f(x, y)$. Which of the points (I)–(VI) lie on the graph of $z = f(x, y)$?

I. $(1, 0, 2)$ II. $(1, 1, 1)$
III. $(0, -1, -2)$ IV. $(-1, 0, -2)$
V. $(0, 1, 1)$ VI. $(-1, -1, 0)$

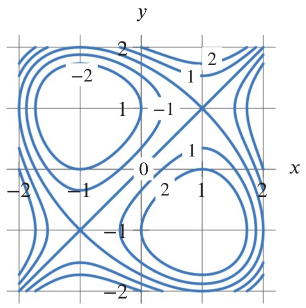

Figure 12.54

18. Match Tables 12.6–12.9 with contour diagrams (I)–(IV) in Figure 12.55.

Table 12.6

$y\backslash x$	-1	0	1
-1	2	1	2
0	1	0	1
1	2	1	2

Table 12.7

$y\backslash x$	-1	0	1
-1	0	1	0
0	1	2	1
1	0	1	0

Table 12.8

$y\backslash x$	-1	0	1
-1	2	0	2
0	2	0	2
1	2	0	2

Table 12.9

$y\backslash x$	-1	0	1
-1	2	2	2
0	0	0	0
1	2	2	2

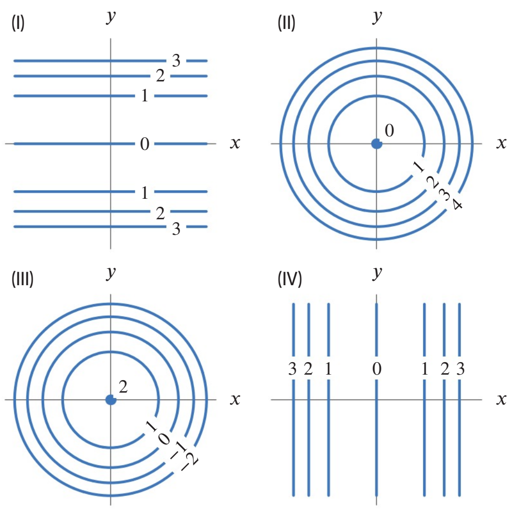

Figure 12.55

The outward unit normal $\vec{n}$ points in the direction of $x\vec{i} + y\vec{j}$, so

$$\vec{n} = \frac{x\vec{i} + y\vec{j}}{\|x\vec{i} + y\vec{j}\|} = \frac{R\cos\theta\vec{i} + R\sin\theta\vec{j}}{R} = \cos\theta\vec{i} + \sin\theta\vec{j}.$$

Therefore, the area vector of the coordinate patch is approximated by

$$\Delta\vec{A} = \vec{n}\,\Delta A \approx \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) R\,\Delta z\,\Delta\theta.$$

Replacing $\Delta\vec{A}$, Δz, and $\Delta\theta$ by $d\vec{A}$, dz, and $d\theta$, we write

$$d\vec{A} = \left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) R\,dz\,d\theta.$$

This gives the following result:

The Flux of a Vector Field Through a Cylinder

The flux of $\vec{F}$ through the cylindrical surface S, of radius R and oriented away from the z-axis, is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R,\theta,z)\cdot\left(\cos\theta\vec{i} + \sin\theta\vec{j}\right) R\,dz\,d\theta,$$

where T is the θz-region corresponding to S.

Example 3 Compute $\int_S \vec{F}\cdot d\vec{A}$ where $\vec{F}(x,y,z) = y\vec{j}$ and S is the part of the cylinder of radius 2 centered on the z-axis with $x \ge 0$, $y \ge 0$, and $0 \le z \le 3$. The surface is oriented toward the z-axis.

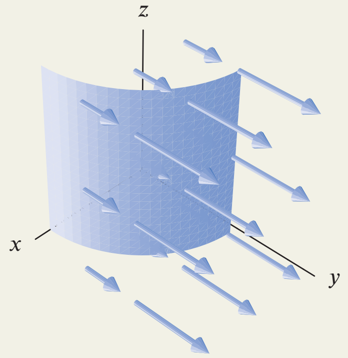

Figure 19.23: The vector field $\vec{F} = y\vec{j}$ on the surface S

Solution In cylindrical coordinates, we have $R = 2$ and $\vec{F} = y\vec{j} = 2\sin\theta\vec{j}$. Since the orientation of S is toward the z-axis, the flux through S is given by

$$\int_S \vec{F}\cdot d\vec{A} = -\int_T 2\sin\theta\vec{j}\cdot(\cos\theta\vec{i} + \sin\theta\vec{j})2\,dz\,d\theta = -4\int_0^{\pi/2}\int_0^3 \sin^2\theta\,dz\,d\theta = -3\pi.$$